PARAMETRIZING FINITE FRAMES AND OPTIMAL FRAME COMPLETIONS

DISSERTATION

Miriam J. Poteet, Civilian

AFIT-ENC-DAM-12-02

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

AFIT-ENC-DAM-12-02

PARAMETRIZING FINITE FRAMES AND OPTIMAL FRAME COMPLETIONS

DISSERTATION

Presented to the Faculty

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

in Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

Miriam J. Poteet, B.S., M.S.

Civilian

December 2012

AFIT-ENC-DAM-12-02

PARAMETRIZING FINITE FRAMES AND OPTIMAL FRAME COMPLETIONS

Miriam J. Poteet, B.S., M.S.
Civilian

Approved:

//signed//	16 November 2012
Matthew Fickus, PhD (Chairman)	Date
//signed//	16 November 2012
Mark E. Oxley, PhD (Member)	Date
//signed//	16 November 2012
Richard K. Martin, PhD (Member)	Date

Accepted:

//signed//	27 November 2012
M. U. Thomas Dean, Graduate School of Engineering and Management	Date

AFIT-ENC-DAM-12-02

Abstract

Frames are used in many signal processing applications. We consider the problem of constructing every frame whose frame operator has a given spectrum and whose vectors have prescribed lengths. For a given spectrum and set of lengths, we know when such a frame exists by the Schur-Horn Theorem; it exists if and only if its spectrum majorizes its squared lengths. We provide a more constructive proof of Horn's original result. This proof is based on a new method for constructing any and all frames whose frame operator has a prescribed spectrum and whose vectors have prescribed lengths. Constructing all such frames requires one choose eigensteps—a sequence of interlacing spectra—which transform the trivial spectrum into the desired one. We give a complete characterization of the convex set of all eigensteps. Taken together, these results permit us, for the first time, to explicitly parametrize the set of all frames whose frame operator has a given spectrum and whose elements have a given set of lengths. Moreover, we generalize this theory to the problem of constructing optimal frame completions. That is, given a preexisting set of measurements, we add new measurements so that the final frame operator has a given spectrum and whose added vectors have prescribed lengths. We introduce a new matrix notation for representing the final spectrum with respect to the initial spectrum and prove that existence of such a frame relies upon a majorization constraint involving the final spectrum and the frame's matrix representation. In a special case, we provide a formula for constructing the optimal frame completion with respect to fusion metrics such as the mean square error (MSE) and frame potential (FP). Such fusion metrics provide a means of evaluating the efficacy of reconstructing signals which have been distorted by noise.

Table of Contents

Page

PARAMETRIZING FINITE FRAMES AND OPTIMAL FRAME COMPLETIONS

I. Introduction

Frames, or redundant sets of vectors in a Hilbert space, have been used in many signal processing applications and are the subject of the research discussed herein. We begin with a few important definitions regarding frames. A sequence of vectors $F = \{f_n\}_{n=1}^N$ is a *frame* in an M-dimensional (real or complex) Hilbert space $\mathbb{H}_M$ if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{N} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$$

for all f in $\mathbb{H}_M$. It is straightforward to show that F is a frame if and only if the F spans $\mathbb{H}_M$, which necessitates $M \leq N$. Letting $\mathbb{K}$ be either the real or complex field, the *synthesis operator* of a sequence of vectors $\{f_n\}_{n=1}^N$ is $F : \mathbb{K}^N \to \mathbb{H}_M$, $Fg := \sum_{n=1}^N g(n)f_n$. The corresponding *frame operator* $FF^* : \mathbb{H}_M \to \mathbb{H}_M$ is given by:

$$FF^*f = \sum_{n=1}^{N} \langle f, f_n\rangle f_n, \tag{1.1}$$

where * denotes the Hermitian transpose of F. Viewing $\mathbb{H}_M$ as $\mathbb{K}^M$, F is the $M \times N$ matrix whose columns are given by the f_n's. The constants A and B are called the *lower* and *upper frame bounds* for F, respectively. Since $\mathbb{H}_M$ is finite-dimensional, then the frame bounds of F are the least and greatest eigenvalues of the frame operator. That is, $\frac{B}{A}$ is the condition number of FF^* and F is a frame if and only if FF^* is invertible.

For a given frame F, the *canonical dual frame* $\tilde{F} = \{\tilde{f}_n\}_{n=1}^N, \tilde{f}_n := (FF^*)^{-1}f_n$, provides a means of expressing any $f \in \mathbb{H}_M$ as a linear combination of frame vectors, that is:

$$f = F\tilde{F}^*f = \sum_{n=1}^{N} \langle f, \tilde{f}_n\rangle f_n. \tag{1.2}$$

The canonical dual frame of F satisfies $F\tilde{F}^* = \mathrm{I}$. Unlike bases, frames provide a means of representing any $f \in \mathbb{H}_M$ in terms of overcomplete sets of vectors. The fact that frames may be redundant make them useful in many signal processing applications where data loss and degradation are concerns [35, 36].

In particular, frames have found a natural application to source coding and robust transmission. Frames have been shown to provide numerically stable reconstruction and resilience to additive noise [27]. While not always fully realistic, additive noise can be used to model channel noise and/or quantization errors which are introduced in analog-to-digital conversion. In addition to noise, erasures of frame coefficients can also be problematic when transmitting a signal [13, 26, 31]. The redundancy of frames allows signals in which data has been lost to be reconstructed in a feasible manner. Such applications have motived current research efforts to find ways of constructing frames with certain desirable properties.

The focus of our research is on constructing finite frames for a given spectrum and set of lengths. This is a generalization of a problem that the field has long been interested in: constructing *unit norm tight frames* (UNTFs). A *tight frame* is one for which $A = B$, and a UNTF is a tight frame with the additional condition that $\|f_n\| = 1$ for all $n = 1, \dots, N$. In this dissertation, we show how to explicitly construct all UNTFs and moreover, how to construct any frame for a given arbitrary spectrum and set of lengths. This theory has also been generalized to the problem of constructing *frame completions*: given a preexisting set of measurements, we add new measurements so that the final frame operator has a given spectrum and whose added vectors have prescribed lengths. More formally, given an initial sequence of vectors $F_N = \{f_n\}_{n=1}^N$ in $\mathbb{H}_M$ whose frame operator has spectrum $\{\alpha_m\}_{m=1}^M$, we wish to complete the frame by adding P additional measurements in order to construct $F = \{f_n\}_{n=1}^{N+P}$ such that $\|f_{N+p}\|^2 = \mu_{N+p}$ for all $p = 1, \dots, P$, and such that the new frame operator of $F = \{f_n\}_{n=1}^{N+P}$ has spectrum $\{\lambda_m\}_{m=1}^M$.

Furthermore, we provide some results on *optimal* frame completions. For real world applications, we often want to choose a frame that is optimal with respect to certain criteria. For example, in cases where part of a signal being transmitted is distorted by noise or lost due to network errors, an optimal frame would be one that minimizes the reconstruction error. Goyal, Kovačević, and Kelner [26] used the *mean square error* (MSE) as a means of evaluating the quality of reconstruction in the former case. The MSE is given by:

$$\text{MSE} = \sigma^2 \text{Tr}[(FF^*)^{-1}] = \sigma^2 \sum_{m=1}^{M} \frac{1}{\lambda_m} \tag{1.3}$$

where $\{\lambda_m\}_{m=1}^M$ are the eigenvalues of FF^* and σ^2 is the variance of the zero-mean, independent, identically distributed added noise $\epsilon = [\epsilon_1\,\epsilon_2\ldots\epsilon_N]^{\mathrm{T}}$. Since calculating the MSE involves inverting the frame operator, which can be difficult, the frame potential may also be used as an alternative notion of the quality of a frame. The *frame potential* (FP) of a sequence $\{f_n\}_{n=1}^N$ in $\mathbb{K}^N$ is given by:

$$\text{FP}(\{f_n\}_{n=1}^N) = \text{Tr}[(FF^*)^2] = \sum_{m=1}^{M} \lambda_m^2 \tag{1.4}$$

where again, $\{\lambda_m\}_{m=1}^M$ are the eigenvalues of FF^*. It has already been shown that a unit norm frame minimizes the MSE and FP if and only if it is tight [5, 26]. However, constructing a UNTF for a given application may not always be achievable if, given an initial frame, one is restricted to adding only a finite number of new measurements. In such cases, determining optimality of a frame is not as straightforward. We provide an algorithm for constructing the optimal frame completion with respect to fusion metrics such as the MSE and FP.

1.1 Major contributions

In this section, we briefly summarize the major contributions that have arisen from our doctoral studies [8, 24, 25]. Our first major contribution (Section 3.1) is Theorem 2 which characterizes and proves the existence of sequences of vectors that generate a given sequence of outer eigensteps. In particular, for any $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$, every sequence of

vectors $\{f_n\}_{n=1}^N$ in $\mathbb{H}_M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^M$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n can be constructed by the two-step process given in Theorem 2, and conversely, any F constructed by this process has $\{\lambda_m\}_{m=1}^M$ as the spectrum of FF^* and $\|f_n\|^2 = \mu_n$ for all n. However, Theorem 2 alone is not an easily implementable algorithm: Step A requires one to choose a sequence of interlacing spectra which transform the trivial spectrum into the desired one; Step B requires one to compute orthonormal eigenbases for each F_n. These steps are addressed in Chapters 4 and 3, respectively.

Our second major contribution (Section 3.2) is to provide a more explicit algorithm for Step B of Theorem 2. Theorem 7 provides a more explicit iterative algorithm for constructing every sequence of vectors $F = \{f_n\}_{n=1}^N$ in $\mathbb{H}_M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^M$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n. The proof of Theorem 7 involves Lagrange interpolating polynomials and a few additional polynomial identities which set it apart from prior theory. Of particular interest is the construction of UNTFs. While there are a few algorithms which can construct UNTFs, little up to this point was known about the manifold of all UNTFs. Two common ways of constructing UNTFs are by (i) truncating a discrete Fourier transform basis [26] and (ii) using an iterative method called *Spectral Tetris* [12]. Our results go far beyond the existing theory in that they not only show how to construct all UNTFs, but they also show how to explicitly construct every frame whose frame operator has a given arbitrary spectrum and whose vectors are of given arbitrary lengths.

The third major contribution (Section 4.2) has been to provide an explicit algorithm, called *Top Kill*, for constructing a sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=0}^N$ which satisfy Definition 12. Given in Theorem 16, this algorithm refines Step A of Theorem 2 by giving an explicit construction of a feasible set of inner eigensteps whenever $\{\lambda_n\}_{n=1}^N$ majorizes $\{\mu_n\}_{n=1}^N$. The Schur-Horn Theorem gives that this set is nonempty if and only if $\{\lambda_n\}_{n=1}^N$ majorizes $\{\mu_n\}_{n=1}^N$. Horn and Johnson [33] already proved that such a sequence exists,

however, their proof is not constructive. The Top Kill algorithm presented not only provides an alternative proof of Horn and Johnson's result, but it provides a method to explicitly construct a sequence of eigensteps as required by Step A of Theorem 2.

Our fourth contribution (Section 4.3) as stated in Theorem 17 gives a complete characterization of the set of all eigensteps for a given $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$. Unlike Top Kill which gives one strategy for constructing one sequence of eigensteps, Theorem 17 provides an algorithm for constructing *all* such sequences. We further show in Chapter 4 that the set of all such sequences is convex. Taken together, these results permit us, for the first time, to explicitly parametrize the set of all frames whose frame operator has a given spectrum and whose elements have a given set of lengths.

The next two major contributions are results of the frame completion problem; that is, given a sequence of vectors $F_N = \{f_n\}_{n=1}^N$, we wish to complete the frame by adding P additional vectors $\{f_{N+p}\}_{p=1}^P$ such that $\|f_{N+p}\|^2 = \mu_{N+p}$ for all $p = 1, \dots P$, and such that the new frame operator of $F = \{f_n\}_{n=1}^{N+P}$ has spectrum $\{\lambda_m\}_{m=1}^M$. In Chapter 5, we determine what it means for a sequence $\{\lambda_m\}_{m=1}^M$ to be (α, μ)-constructible (Theorem 39) and show that constructibility relies upon a majorization constraint involving $\{\lambda_m\}_{m=1}^M$ and the frame's spectral partition matrix. The second major contribution in Chapter 5 is a new algorithm called *Chop Kill* (Theorem 40) which is a generalization of the Top Kill algorithm in Chapter 4. This algorithm enables us to explicitly construct a valid sequence of continued outer eigensteps whenever $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^M$ majorizes $\{\mu_{N+p}\}_{p=1}^P$, i.e., for any (α, μ)-constructible sequence. Having a valid sequence of eigensteps, we then return to Step B of the two-step process and use the algorithm given in Theorem 7 in order to explicitly construct the added frame vectors $\{f_{N+p}\}_{p=1}^P$.

Finally, our last major contribution (Section 6.2) as stated in Theorem 48, is a closed-form solution for the optimal (α, μ)-constructible sequence in the special case that $\{\mu_{N+p}\}_{p=1}^P$ are all of equal lengths. To prove this result, we measure optimality with respect to

majorization, and show that the optimal (α, μ)-constructible sequence is the the one which is majorized by all other sequences which can be built from $\{\alpha_m\}_{m=1}^M$ and $\{\mu_{N+p}\}_{p=1}^P$, i.e., all other (α, μ)-constructible sequences.

The following is a list of current publications that have arisen from my PhD studies:

Journal Articles

1. Cahill, Jameson, Matthew Fickus, Dustin G. Mixon, Miriam J. Poteet and Nate Strawn. "Constructing finite frames of a given spectrum and set of lengths," to appear in *Applied and Computational Harmonic Analysis*, 22 pages (2012).

2. Fickus, Matthew, Dustin G. Mixon, Miriam J. Poteet and Nate Strawn. "Constructing all self-adjoint matrices with prescribed spectrum and diagonal," submitted to: *Advances in Computational Mathematics*, 20 pages.

Book Chapters

1. Fickus, Matthew, Dustin G. Mixon and Miriam J. Poteet. "Constructing all finite frames with a given spectrum," in *Finite Frames: Theory and Applications*, P. G. Casazza and G. Kutyniok eds., Birkhauser, pp. 55–107 (2012).

Conference Proceedings

1. Fickus, Matthew, Dustin G. Mixon and Miriam J. Poteet. "Frame completions for optimally robust reconstruction," *Proceedings of SPIE*, 8138: 81380Q/1-8 (2011).

1.2 Outline

In Chapter 2, we review the relevant background literature of frames, majorization, and interlacing. In Chapter 3, we present one of our main result (Theorem 2) which characterizes and proves the existence of sequences of vectors that generate a given sequence of outer eigensteps. We then present a second major result (Theorem 7) which provides a more explicit iterative algorithm for constructing every sequence of

vectors $F = \{f_n\}_{n=1}^N$ in $\mathbb{H}_M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^M$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n. Chapter 4 contains the Top Kill algorithm and a result (Theorem 17) which gives a complete characterization of the set of all eigensteps, as needed by Step A of Theorem 2. Finally, we discuss frame completions in Chapters 5 and 6. Specifically, Chapter 5 contains the Chop Kill algorithm and a result (Theorem 39) which determines when a sequence is (α, μ)-constructible. Chapter 6 contains our final major result (Theorem 48) which is an explicit formula for the optimal (α, μ)-constructible sequence in the special case that $\{\mu_{N+p}\}_{p=1}^P$ are all of equal lengths.

II. Literature Review

In this chapter, we review the literature that is relevant to our research. In Section 2.1, we discuss the relevant frame theory literature. Simply stated, frames are redundant sets of vectors in a Hilbert space. In contrast to unique representation of vectors with respect to a given basis, frames provide a means of representing any f in $\mathbb{H}_M$ in terms of overcomplete sets of vectors. Frames have many useful applications including, but not limited to, their resiliency to additive noise and quantization [27], and erasures [26].

In Section 2.2, we discuss literature relevant to majorization. Namely, given two nonnegative nonincreasing sequences $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$, we say that $\{\lambda_n\}_{n=1}^N$ *majorizes* $\{\mu_n\}_{n=1}^N$, denoted $\{\lambda_n\}_{n=1}^N \succeq \{\mu_n\}_{n=1}^N$, if

$$\begin{aligned}\sum_{n'=1}^{n} \lambda_{n'} &\geq \sum_{n'=1}^{n} \mu_{n'} \qquad \forall n = 1, \ldots, N-1,\\ \sum_{n'=1}^{N} \lambda_{n'} &= \sum_{n'=1}^{N} \mu_{n'}.\end{aligned}$$

Hardy *et al.* discussed many of the main ideas of majorization in their 1934 book *Inequalities* [29]. From that point forward, the concepts of majorization have been applied to a broad spectrum of fields from statistical mechanics to graph theory [2].

Also in Section 2.2, we discuss the background literature of interlacing. We say a nonnegative nonincreasing sequence $\{\gamma_m\}_{m=1}^M$ *interlaces* on another such sequence $\{\beta_m\}_{m=1}^M$, denoted $\{\beta_m\}_{m=1}^M \sqsubseteq \{\gamma_m\}_{m=1}^M$, provided

$$\beta_M \leq \gamma_M \leq \beta_{M-1} \leq \gamma_{M-1} \leq \cdots \leq \beta_2 \leq \gamma_2 \leq \beta_1 \leq \gamma_1.$$

Under the convention $\gamma_{M+1} := 0$, we have that $\{\beta_m\}_{m=1}^M \sqsubseteq \{\gamma_m\}_{m=1}^M$ if and only if $\gamma_{m+1} \leq \beta_m \leq \gamma_m$ for all $m = 1, \ldots, M$. In the context of our research, interlacing is applied to the spectra of partial sums of the frame operator, and is also useful in that it provides an alternative way of viewing majorization [33].

2.1 Frames

Frame theory was first introduced by Duffin and Schaeffer in 1952 in their paper on nonharmonic Fourier series [20]. The fact that frames may be redundant makes them useful in many signal processing applications where data loss and degradation are a concern [35, 36]. In particular, quantization error is introduced in analog-to-digital conversion, that is, when a signal that is varying continuously is restricted to a set of discrete values. There, round-off and truncation errors may distort the signal. While not always realistic, additive noise can be used to model channel noise and/or quantization errors. Goyal *et al.* found that consistent reconstruction methods yielded smaller reconstruction errors than that of classical methods and that UNTFs are optimally robust with respect to additive noise [26, 27].

Another application of frames is that of data transmission. For example, when data is transmitted over a communications network such as the internet, it is possible that the information being transmitted can be distorted or lost due to network errors. When modeled in terms of frames, this corresponds to *erasures*, namely having some of the transform coefficients are lost during the transmission. Erasures of frame coefficients have been well studied [13, 26, 31]. In particular, it has been shown that UNTFs are useful for data transmission which is robust against erasures [13]. One useful class of UNTFs in this setting are *harmonic tight frames* (HTFs) which are constructed from a discrete Fourier transform basis [13].

In light of these applications, we often want to choose a frame that is optimal with respect to certain criteria. The authors of [5, 26, 27] have already explored metrics such as the MSE (1.3) and FP (1.4) as a means of quantifying reconstruction errors in the applications mentioned above, and have shown that unit norm frames minimize these quantities if and only if they are tight [5, 26]. In particular, for an $M \times N$ UNTF the minimum value of the FP is precisely $\frac{N^2}{M}$ [5]. In regards to channel noise and quantization

errors, consistent reconstruction methods were shown to be optimal, with the MSE of the reconstruction being equal to $\frac{(M\sigma)^2}{N}$ where σ^2 is the variance of the added noise [27]. UNTFs are also known as *Welch-Bound-Equality sequences* since they give equality to the *Welch bound*, that is $\mathrm{Tr}[(FF^*)^2] = \frac{N^2}{M}$ [47, 48]. These sequences are shown to be the optimal signature sequences in CDMA systems [46].

When considering frames for applications such as those mentioned above, we often wish to retain control over the spectrum of FF^* and the lengths of the frame vectors [24]. Of particular interest is the construction of UNTFs. It is known that the manifold of all real $M \times N$ UNTFs, modulo rotations, has dimension $(N-M-1)(M-1)$ when $N > M$ [22]. When $M = N + 1$, it is shown in [26], that such tight frames are essentially unique. Additionally, it is shown in [43] that explicit local parameterizations of this manifold can be constructed. Since the columns of F must have unit norm and the rows must be orthogonal, constructing a UNTF is not an easy task as it involves solving a large system of quadratic equations. In fact, only a few algorithms exist which can construct examples of such frames, and even then, they only construct a finite subset of this continuous manifold. Two common ways of constructing UNTFs are HTFs [26] and iteratively by a method called Spectral Tetris [12]. Tight frames can also be constructed via a method based on alternating projections as discussed in [45] and by using Householder transformations [23].

Algorithms constructing UNTFs have also been proposed to address the *Paulsen problem*: How close is a frame which is almost tight and almost unit norm to some UNTF? This question arises in applications when given an initial frame, it may be more desirable to transform the given frame into a UNTF which, as described above, is optimal for a variety of reasons. It is shown in [11] that for a given unit norm frame, a nearby UNTF can iteratively be found by a method of gradient descent of the frame potential. Additionally, Bodmann and Casazza [6] have shown that equal-norm Parseval frames (frames for which

$A = B = 1$) can be constructed via a method based on a system of ordinary differential equations.

Tight frames can also be constructed from an existing set of vectors. That is, given an initial sequence of vectors, additional vectors with prescribed norms are added in order to construct a tight frame. The necessary and sufficient conditions for the existence of such frames can be found in [39]. In [23], the problem of tight frame completions is considered as well as finding a lower bound for the number of vectors to add in order to construct a tight frame. However, as the authors of [23] point out, when the added vectors are required to be unit norm, the lower bound they propose is not a true lower bound. Massey and Ruiz [39] considered this problem as well, and calculate the minimum number of vectors to add regardless of whether they are unit norm or not. Additionally, they provide an algorithm for constructing a tight frame completion from an initial sequence of vectors.

For real world applications, it may be desirable to complete the frame in such a way that it is optimal with respect to certain criteria. For example, in cases where part of a signal being transmitted is distorted by noise or lost due to network errors, an optimal frame would be one that minimizes the reconstruction error. Constructing a UNTF for a given application may not always be achievable if one is restricted to adding only a finite number of new vectors in order to complete the frame. Moreover, in this setting, it is not straightforward to *optimally* complete the frame. Massey, Ruiz, and Stojanoff show in [40] and [41] that determining optimality of a frame completion does not have to be limited to the MSE and the FP, but rather can be measured with respect to majorization. In short, they show that optimal frame completions are minimizers of a family of convex functionals, that includes, but is not limited to the MSE and FP. Algorithms for solving for the optimal frame completion can be found in [40] and [41]; however, as the authors note in [41], the algorithm's performance in cases when a larger number of vectors is added or when the

dimension of the space is large is contingent upon a conjecture which they have yet to prove.

As we will show in the chapters that follow, for the first time, we have been able to go beyond the existing theory of constructing UNTFs by showing how to explicitly construct every frame belonging to the manifold of all real UNTFs. Moreover, we provide a partial solution to the frame completion problem we posed in [24] and present an algorithm for finding the optimal frame completion in a special case which can be implemented in a finite number of iterations. Being able to explicitly construct every frame whose frame operator has a given arbitrary spectrum and whose vectors are of given arbitrary lengths requires existing theory of majorization and the Schur-Horn Theorem which are discussed in the following section. Relevant interlacing literature is also discussed.

2.2 Majorization and interlacing

Majorization arises in a variety of contexts including combinatorial analysis, matrix theory, numerical analysis, and statistics just to name a few [2, 29]. Here, we are interested in how the majorization inequalities apply to matrix theory. Specifically, the concept of majorization was used by Schur in [42] where he showed that the spectrum of a self-adjoint positive semidefinite matrix necessarily majorizes its diagonal entries. Some time later, Horn [32] proved the converse: if $\{\lambda_n\}_{n=1}^N \succeq \{\mu_n\}_{n=1}^N$, then there exists a self-adjoint matrix that has $\{\lambda_n\}_{n=1}^N$ as its spectrum and $\{\mu_n\}_{n=1}^N$ as its diagonal [32]. Combining these two results gives the Schur-Horn Theorem which states that there exists a positive semidefinite self-adjoint matrix with spectrum $\{\lambda_n\}_{n=1}^N$ and diagonal entries $\{\mu_n\}_{n=1}^N$ if and only if $\{\lambda_n\}_{n=1}^N \succeq \{\mu_n\}_{n=1}^N$. Alternative proofs of this theorem are given in [37].

Existence of frames with predefined lengths have been studied in [10, 21]. In regards to tight frames, Casazza *et al.* gave necessary conditions upon a set of lengths $\{\mu_n\}_{n=1}^N$ to prove existence of tight frames for which $\|f_n\| = \mu_n$ for all n. Moreover, it is shown that for any set of lengths $\{\mu_n\}_{n=1}^N$ which satisfy the *fundamental inequality*, $\max_{n=1,\dots,N}(\mu_n^2) \leq$

$\frac{1}{M}\sum_{n=1}^{N}\mu_n^2$, there exists a tight frame for which $\|f_n\| = \mu_n$ for all n. Viswanath and Anantharam [46] independently discovered the fundamental inequality when constructing optimal signature sequences for CDMA systems. The connection between constructing optimal CDMA signature sequences and the Schur-Horn Theorem is discussed in [44], and finite-step algorithms for constructing such sequences are given.

In context of our research, by applying the Schur-Horn Theorem to the *Gram matrix* F^*F, whose diagonal entries are given by $\{\|f_n\|^2\}_{n=1}^N$ and whose spectrum is $\{\lambda_n\}_{n=1}^N$, we are able to determine when a frame with prescribed spectrum and lengths exists. Here it is important to note that the spectrum of the Gram matrix F^*F is a zero-padded version of the spectrum $\{\lambda_m\}_{m=1}^M$ of the frame operator FF^*. In [1], Antezana *et al.* make the connection between the Schur-Horn Theorem and majorization explicit, extending the Schur-Horn Theorem to give conditions on the existence of frames with prescribed norms and frame operator FF^*. This connection is also made clear in [45]. Majorization also plays a role in the frame completion problem and is used to determine which frames can be built from a preexisting set of vectors.

Also of interest is the problem of explicitly constructing frames which satisfy the Schur-Horn Theorem. Algorithms relying on Givens rotations [14, 15, 18, 19] have been used to produce self-adjoint matrices with a given majorized diagonal. Dhillon *et al.*, in particular, improve upon the Chan-Li [15] and Bendel-Mickey [4] algorithms and present more generalized versions of these algorithms which have the ability of constructing a much larger subset of the corresponding manifold of desired matrices. Chu [16] also considered constructing such matrices via a lift-and-projection method and a projected gradient method. Each of these methods provides a more constructive proof of Horn's original result [32]. In the remainder of this section, we discuss interlacing and its relevance to majorization.

Interlacing inequalities arise when considering the relationship among eigenvalues of principle submatrices [34, 49] and have also been applied to matrices associated with graphs [28]. Interlacing is also useful in that it provides an alternative way of viewing majorization [33]. Here we consider the inverse eigenvalue problem which involves reconstructing a matrix from prescribed spectral data. Inverse eigenvalue problems not only pose spectral constraints, but also structural constraints upon a solution matrix. A survey of how interlacing applies to these types of problems can be found in [7], and more recently in [17].

For our research, interlacing comes into play by considering partial sums of the frame operator (3.4). First let $\{\lambda_{n;m}\}_{m=1}^{M}$ and $\{\lambda_{n+1;m}\}_{m=1}^{M}$ denote the spectrum of $F_nF_n^*$ and $F_{n+1}F_{n+1}^*$, respectively. Here, F_n denotes the $M\times n$ submatrix of the $M\times N$ matrix F obtained by taking its first n columns. By applying a classical result in [33] to $F_{n+1}F_{n+1}^* = F_nF_n^* + f_{n+1}f_{n+1}^*$, where $f_{n+1}f_{n+1}^*$ is a rank-one positive operator, it can be shown that $\{\lambda_{n+1;m}\}_{m=1}^{M}$ interlaces with $\{\lambda_{n;m}\}_{m=1}^{M}$. Another result of Horn and Johnson, which will be revisited in Chapters 3 and 4, states that given sequences $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$ such that $\{\lambda_n\}_{n=1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$, there exists a sequence $\{\tilde{\lambda}_n\}_{n=1}^{N-1}$ such that $\{\lambda_n\}_{n=1}^{N}$ interlaces with $\{\tilde{\lambda}_n\}_{n=1}^{N-1}$ and $\{\tilde{\lambda}_n\}_{n=1}^{N-1}$ majorizes $\{\mu_n\}_{n=1}^{N-1}$ [33].

III. Constructing frames of a given spectrum and set of lengths

In this chapter, we provide an algorithm for how to explicitly construct every frame whose frame operator has a given arbitrary spectrum and whose vectors are of given arbitrary lengths. The major results are Theorem 2, which gives a two-step algorithm for constructing all such frames, and Theorem 7, which makes Step B of Theorem 2 much more explicit and implementable.

In order to construct all such frames, we make use of the existing theory of majorization and the Schur-Horn Theorem. Namely, given two nonnegative nonincreasing sequences $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$, recall that $\{\lambda_n\}_{n=1}^N$ majorizes $\{\mu_n\}_{n=1}^N$, denoted $\{\lambda_n\}_{n=1}^N \succeq \{\mu_n\}_{n=1}^N$, if

$$\sum_{n'=1}^{n} \lambda_{n'} \geq \sum_{n'=1}^{n} \mu_{n'} \qquad \forall n = 1, \ldots, N-1, \tag{3.1}$$

$$\sum_{n'=1}^{N} \lambda_{n'} = \sum_{n'=1}^{N} \mu_{n'}. \tag{3.2}$$

One of Schur's results stated that the spectrum of a self-adjoint positive semidefinite matrix necessarily majorizes its diagonal entries [42]. Some time later, Horn showed that if $\{\lambda_n\}_{n=1}^N \succeq \{\mu_n\}_{n=1}^N$, then there exists a self-adjoint matrix that has $\{\lambda_n\}_{n=1}^N$ as its spectrum and $\{\mu_n\}_{n=1}^N$ as its diagonal [32]. Combining these two results gives the Schur-Horn Theorem:

Schur-Horn Theorem. *There exists a positive semidefinite self-adjoint matrix with spectrum $\{\lambda_n\}_{n=1}^N$ and diagonal entries $\{\mu_n\}_{n=1}^N$ if and only if $\{\lambda_n\}_{n=1}^N \succeq \{\mu_n\}_{n=1}^N$.*

By applying this theorem to the Gram matrix F^*F, whose diagonal entries are given by $\{\|f_n\|^2\}_{n=1}^N$ and whose spectrum is $\{\lambda_n\}_{n=1}^N$, we are able to determine when a frame with prescribed spectrum and lengths exists. Here it is important to note that the spectrum of the Gram matrix is a zero-padded version of the spectrum $\{\lambda_m\}_{m=1}^M$ of the frame operator FF^*. The only difference between the eigenvalues of the Gram matrix and the frame operator is zero eigenvalues; that is, $\lambda_n = 0$ for $M < n \leq N$.

For the current research, we consider an alternative approach to majorization which involves the repeated application of eigenvalue interlacing [33]. Recall that we say a nonnegative nonincreasing sequence $\{\gamma_m\}_{m=1}^M$ interlaces on another such sequence $\{\beta_m\}_{m=1}^M$, denoted $\{\beta_m\}_{m=1}^M \sqsubseteq \{\gamma_m\}_{m=1}^M$, provided

$$\beta_M \leq \gamma_M \leq \beta_{M-1} \leq \gamma_{M-1} \leq \cdots \leq \beta_2 \leq \gamma_2 \leq \beta_1 \leq \gamma_1. \tag{3.3}$$

Interlacing arises in frame theory in the following context: given any sequence of vectors $F = \{f_n\}_{n=1}^N$ in $\mathbb{H}_M$, then for every $n = 1, \ldots, N$, we consider the partial sequence of vectors $F_n := \{f_{n'}\}_{n'=1}^n$. Note that $F_N = F$ and the frame operator of F_n is

$$F_n F_n^* = \sum_{n'=1}^{n} f_{n'} f_{n'}^*. \tag{3.4}$$

Let $\{\lambda_{n;m}\}_{m=1}^M$ denote the spectrum of (3.4). For any $n = 1, \ldots, N-1$, (3.4) gives that $F_{n+1}F_{n+1}^* = F_n F_n^* + f_{n+1}f_{n+1}^*$. By a result in [33] involving the addition of rank-one positive operators we know that $\{\lambda_{n;m}\}_{m=1}^M \sqsubseteq \{\lambda_{n+1;m}\}_{m=1}^M$. That is, the spectrum of the $(n+1)$st partial sum interlaces on the spectrum of the nth partial sum. Moreover, if $\|f_n\|^2 = \mu_n$ for all $n = 1, \ldots, N$, then for any such n,

$$\sum_{m=1}^{M} \lambda_{n;m} = \mathrm{Tr}(F_n F_n^*) = \mathrm{Tr}(F_n^* F_n) = \sum_{n'=1}^{n} \|f_{n'}\|^2 = \sum_{n'=1}^{n} \mu_{n'}. \tag{3.5}$$

Thus, for any given frame, the sequence of the spectra of its partial frame operators satisfies a set of interlacing and trace conditions. As such, in order to construct a frame F with prescribed spectrum and lengths, we give a name to sequences of interlacing spectra that satisfy (3.5):

Definition 1. Given nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$, a sequence of *outer eigensteps* is a doubly-indexed sequence of sequences $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ for which:

(i) The initial sequence is trivial: $\lambda_{0;m} = 0$, $\forall m = 1, \ldots, M$.

(ii) The final sequence is $\{\lambda_m\}_{m=1}^M$: $\lambda_{N;m} = \lambda_m$, $\forall m = 1, \ldots, M$.

(iii) The sequences interlace: $\{\lambda_{n-1;m}\}_{m=1}^{M} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^{M}$, $\forall n = 1, \ldots, N$.

(iv) The trace condition is satisfied: $\sum_{m=1}^{M} \lambda_{n;m} = \sum_{n'=1}^{n} \mu_{n'}$, $\forall n = 1, \ldots, N$.

We refer to the values $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ as outer eigensteps since they arise from sums of outer products of the f_n's (frame operators). Every sequence of vectors whose frame operator has the spectrum $\{\lambda_m\}_{m=1}^{M}$ and whose vectors have squared lengths $\{\mu_n\}_{n=1}^{N}$ generates a sequence of outer eigensteps. In the remainder of this chapter, we show that the converse is also true. To be precise, one of our main results, Theorem 2, provides an algorithm for explicitly constructing every possible finite frame of a given spectrum and set of lengths, in terms of such eigensteps.

3.1 The necessity and sufficiency of eigensteps

In this section, our goal is to prove the following theorem:

Theorem 2. *For any nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$, every sequence of vectors $F = \{f_n\}_{n=1}^{N}$ in $\mathbb{H}_M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^{M}$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n can be constructed by the following process:*

A. Pick outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ as in Definition 1.

B. For each $n = 1, \ldots, N$, consider the polynomial:

$$p_n(x) := \prod_{m=1}^{M}(x - \lambda_{n;m}). \tag{3.6}$$

Take any $f_1 \in \mathbb{H}_M$ such that $\|f_1\|^2 = \mu_1$. For each $n = 1, \ldots, N-1$, choose any f_{n+1} such that

$$\|P_{n;\lambda} f_{n+1}\|^2 = -\lim_{x \to \lambda}(x - \lambda)\frac{p_{n+1}(x)}{p_n(x)} \tag{3.7}$$

for all $\lambda \in \{\lambda_{n;m}\}_{m=1}^{M}$, where $P_{n;\lambda}$ denotes the orthogonal projection operator onto the eigenspace $\mathrm{N}(\lambda\mathrm{I} - F_n F_n^*)$ of the frame operator $F_n F_n^*$ of $F_n = \{f_{n'}\}_{n'=1}^{n}$. The limit in (3.7) *exists and is nonpositive.*

Conversely, any F constructed by this process has $\{\lambda_m\}_{m=1}^M$ as the spectrum of FF^* and $\|f_n\|^2 = \mu_n$ for all n. Moreover, for any F constructed in this manner, the spectrum of $F_nF_n^*$ is $\{\lambda_{n;m}\}_{m=1}^M$ for all $n = 1,\dots,N$.

Theorem 2 is not an easily-implementable algorithm nor does it address the existence of such an F for a given $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$. Later in this chapter, we will address Step B and how to compute orthonormal eigenbases for each F_n. In the following chapter, we discuss Step A which requires one choose a valid sequence of eigensteps. In order to prove Theorem 2, we first prove the following lemma:

Lemma 3. *Let $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ be nonnegative and nonincreasing, and let $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ be any corresponding sequence of outer eigensteps as in Definition 1. If a sequence of vectors $F = \{f_n\}_{n=1}^N$ has the property that the spectrum of the frame operator $F_nF_n^*$ of $F_n = \{f_{n'}\}_{n'=1}^n$ is $\{\lambda_{n;m}\}_{m=1}^M$ for all $n = 1,\dots,N$, then the spectrum of FF^* is $\{\lambda_m\}_{m=1}^M$ and $\|f_n\|^2 = \mu_n$ for all $n = 1,\dots,N$.*

Proof. Definition 1.(ii) immediately gives that the spectrum of $FF^* = F_NF_N^*$ is indeed $\{\lambda_m\}_{m=1}^M = \{\lambda_{N;m}\}_{m=1}^M$, as claimed. Moreover, for any $n = 1,\dots,N$, Definition 1.(iv) gives

$$\sum_{n'=1}^n \|f_{n'}\|^2 = \mathrm{Tr}(F_n^*F_n) = \mathrm{Tr}(F_nF_n^*) = \sum_{m=1}^M \lambda_{n;m} = \sum_{n'=1}^n \mu_{n'}. \tag{3.8}$$

Letting $n = 1$ in (3.8) gives $\|f_1\|^2 = \mu_1$, while for $n = 2,\dots,N$, considering (3.8) at both n and $n-1$ gives

$$\|f_n\|^2 = \sum_{n'=1}^n \|f_{n'}\|^2 - \sum_{n'=1}^{n-1} \|f_{n'}\|^2 = \sum_{n'=1}^n \mu_{n'} - \sum_{n'=1}^{n-1} \mu_{n'} = \mu_n. \qquad \square$$

The next result gives conditions that a vector must satisfy in order for it to perturb the spectrum of a given frame operator in a desired way, and was inspired by the proof of the Matrix Determinant Lemma and its application in [3].

Theorem 4. *Let $F_n = \{f_{n'}\}_{n'=1}^n$ be an arbitrary sequence of vectors in $\mathbb{H}_M$ and let $\{\lambda_{n;m}\}_{m=1}^M$ denote the eigenvalues of the corresponding frame operator $F_nF_n^*$. For any choice of f_{n+1}*

in $\mathbb{H}_M$*, let* $F_{n+1} = \{f_{n'}\}_{n'=1}^{n+1}$*. Then for any* $\lambda \in \{\lambda_{n;m}\}_{m=1}^{M}$*, the norm of the projection of* f_{n+1} *onto the eigenspace* $\mathrm{N}(\lambda\mathrm{I} - F_nF_n^*)$ *is given by*

$$\|P_{n;\lambda}f_{n+1}\|^2 = -\lim_{x\to\lambda}(x-\lambda)\frac{p_{n+1}(x)}{p_n(x)},$$

where $p_n(x)$ *and* $p_{n+1}(x)$ *denote the characteristic polynomials of* $F_nF_n^*$ *and* $F_{n+1}F_{n+1}^*$*, respectively.*

Proof. For the sake of notational simplicity, we let $F := F_n$, $f := f_{n+1}$ and so $F_{n+1}F_{n+1}^* = FF^* + ff^*$. Suppose x is not an eigenvalue of $F_{n+1}F_{n+1}^*$. Then:

$$\begin{aligned} p_{n+1}(x) &= \det(x\mathrm{I} - FF^* - ff^*) \\ &= \det(x\mathrm{I} - FF^*)\det(\mathrm{I} - (x\mathrm{I} - FF^*)^{-1}ff^*) \\ &= p_n(x)\det(\mathrm{I} - (x\mathrm{I} - FF^*)^{-1}ff^*). \end{aligned} \tag{3.9}$$

We can simplify the determinant of $\mathrm{I} - (x\mathrm{I} - FF^*)^{-1}ff^*$ by multiplying by certain matrices with unit determinant:

$$\begin{aligned} &\det(\mathrm{I} - (x\mathrm{I} - FF^*)^{-1}ff^*) \\ &\quad= \det\left(\begin{bmatrix}\mathrm{I} & 0\\ f^* & 1\end{bmatrix}\begin{bmatrix}\mathrm{I} - (x\mathrm{I} - FF^*)^{-1}ff^* & -(x\mathrm{I} - FF^*)^{-1}f\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathrm{I} & 0\\ -f^* & 1\end{bmatrix}\right) \\ &\quad= \det\left(\begin{bmatrix}\mathrm{I} & 0\\ f^* & 1\end{bmatrix}\begin{bmatrix}\mathrm{I} & -(x\mathrm{I} - FF^*)^{-1}f\\ -f^* & 1\end{bmatrix}\right) \\ &\quad= \det\left(\begin{bmatrix}\mathrm{I} & -(x\mathrm{I} - FF^*)^{-1}f\\ 0 & 1 - f^*(x\mathrm{I} - FF^*)^{-1}f\end{bmatrix}\right) \\ &\quad= 1 - f^*(x\mathrm{I} - FF^*)^{-1}f. \end{aligned} \tag{3.10}$$

We now use (3.9) and (3.10) with the spectral decomposition $FF^* = \sum_{m=1}^{M}\lambda_{n;m}u_mu_m^*$:

$$p_{n+1}(x) = p_n(x)(1 - f^*(x\mathrm{I} - FF^*)^{-1}f) = p_n(x)\left(1 - \sum_{m=1}^{M}\frac{|\langle f, u_m\rangle|^2}{x - \lambda_{n;m}}\right). \tag{3.11}$$

Rearranging (3.11) and grouping the eigenvalues $\Lambda = \{\lambda_{n;m}\}_{m=1}^{M}$ according to multiplicity then gives

$$\frac{p_{n+1}(x)}{p_n(x)} = 1 - \sum_{m=1}^{M} \frac{|\langle f, u_m\rangle|^2}{x - \lambda_{n;m}} = 1 - \sum_{\lambda'\in\Lambda} \frac{\|P_{n;\lambda'} f\|^2}{x - \lambda'} \qquad \forall x \notin \Lambda.$$

As such, for any $\lambda \in \Lambda$,

$$\begin{aligned}
\lim_{x\to\lambda}(x-\lambda)\frac{p_{n+1}(x)}{p_n(x)} &= \lim_{x\to\lambda}(x-\lambda)\Big(1 - \sum_{\lambda'\in\Lambda}\frac{\|P_{n;\lambda'} f\|^2}{x-\lambda'}\Big)\\
&= \lim_{x\to\lambda}\Big[(x-\lambda) - \|P_{n;\lambda} f\|^2 - \sum_{\lambda'\neq\lambda}\|P_{n;\lambda'} f\|^2 \frac{x-\lambda}{x-\lambda'}\Big]\\
&= -\|P_{n;\lambda} f\|^2,
\end{aligned}$$

yielding our claim. □

While the following two lemmas depend only on basic algebra and calculus, their results will be used in the proof of Theorem 2.

Lemma 5. *If $\{\beta_m\}_{m=1}^{M}$ and $\{\gamma_m\}_{m=1}^{M}$ are real and nonincreasing, then $\{\beta_m\}_{m=1}^{M} \sqsubseteq \{\gamma_m\}_{m=1}^{M}$ if and only if*

$$\lim_{x\to\beta_m}(x-\beta_m)\frac{q(x)}{p(x)} \leq 0 \qquad \forall m = 1, \dots, M,$$

where $p(x) = \prod_{m=1}^{M}(x-\beta_m)$ and $q(x) = \prod_{m=1}^{M}(x-\gamma_m)$.

Proof. ($\Rightarrow$) Let $\{\gamma_m\}_{m=1}^{M}$ interlace on $\{\beta_m\}_{m=1}^{M}$, and let $\lambda = \beta_m$ for some $m = 1, \dots, M$. Letting L_p denote the multiplicity of λ as a root of $p(x)$, the fact that $\{\beta_m\}_{m=1}^{M} \sqsubseteq \{\gamma_m\}_{m=1}^{M}$ implies that the multiplicity L_q of λ as a root of $q(x)$ is at least $L_p - 1$. Moreover, if $L_q > L_p - 1$ then our claim holds at λ since

$$\lim_{x\to\lambda}(x-\lambda)\frac{q(x)}{p(x)} = 0 \leq 0.$$

Meanwhile, if $L_q = L_p - 1$, then choosing $m_p = \min\{m : \beta_m = \lambda\}$ gives

$$\begin{array}{llcccl}
\beta_m > \lambda, & 1 & \leq & m & \leq & m_p - 1,\\
\beta_m = \lambda, & m_p & \leq & m & \leq & m_p + L_p - 1,\\
\beta_m < \lambda, & m_p + L_p & \leq & m & \leq & M.
\end{array} \tag{3.12}$$

We now determine a similar set of relations between λ and all choices of γ_m. For $1 \leq m \leq m_p - 1$, interlacing and (3.12) imply $\gamma_m \geq \beta_m > \lambda$. If instead $m_p + 1 \leq m \leq m_p + L_p - 1$, then interlacing and (3.12) imply $\lambda = \beta_m \leq \gamma_m \leq \beta_{m-1} = \lambda$ and so $\gamma_m = \lambda$. Another possibility is to have $m_p + L_p + 1 \leq m \leq M$, in which case interlacing and (3.12) imply $\gamma_m \leq \beta_{m-1} < \lambda$. Taken together, we have

$$\begin{aligned}
\gamma_m &> \lambda, & 1 &\leq m \leq m_p - 1, \\
\gamma_m &= \lambda, & m_p + 1 &\leq m \leq m_p + L_p - 1, \\
\gamma_m &< \lambda, & m_p + L_p + 1 &\leq m \leq M.
\end{aligned} \tag{3.13}$$

Note that (3.13) is unlike (3.12) in that in (3.13), the relationship between γ_m and λ is still undecided for $m = m_p$ and $m = m_p + L_p$. Indeed, in general we only know $\gamma_{m_p} \geq \beta_{m_p} = \lambda$, and so either $\gamma_{m_p} = \lambda$ or $\gamma_{m_p} > \lambda$. Similarly, we only know $\gamma_{m_p+L_p} \leq \beta_{m_p+L_p-1} = \lambda$, so either $\gamma_{m_p+L_p} = \lambda$ or $\gamma_{m_p+L_p} < \lambda$. Of these four possibilities, three lead to either having $L_q = L_p + 1$ or $L_q = L_p$; only the case where $\gamma_{m_p} > \lambda$ and $\gamma_{m_p+L_p} < \lambda$ leads to our current assumption that $L_q = L_p - 1$. As such, under this assumption (3.13) becomes

$$\begin{aligned}
\gamma_m &> \lambda, & 1 &\leq m \leq m_p, \\
\gamma_m &= \lambda, & m_p + 1 &\leq m \leq m_p + L_p - 1, \\
\gamma_m &< \lambda, & m_p + L_p &\leq m \leq M.
\end{aligned} \tag{3.14}$$

We now prove our claim using (3.12) and (3.14):

$$\begin{aligned}
\lim_{x\to\lambda}(x-\lambda)\frac{q(x)}{p(x)} &= \lim_{x\to\lambda} \frac{(x-\lambda)^{L_p}\prod_{m=1}^{m_p}(x-\gamma_m)\prod_{m=m_p+L_p}^{M}(x-\gamma_m)}{(x-\lambda)^{L_p}\prod_{m=1}^{m_p-1}(x-\beta_m)\prod_{m=m_p+L_p}^{M}(x-\beta_m)} \\
&= \frac{\prod_{m=1}^{m_p}(\lambda-\gamma_m)\prod_{m=m_p+L_p}^{M}(\lambda-\gamma_m)}{\prod_{m=1}^{m_p-1}(\lambda-\beta_m)\prod_{m=m_p+L_p}^{M}(\lambda-\beta_m)} \\
&< 0.
\end{aligned}$$

($\Leftarrow$) We prove by induction on M. For $M = 1$, we have $p(x) = x - \beta_1$ and $q(x) = x - \gamma_1$, and so if

$$0 \geq \lim_{x \to \beta_1} (x - \beta_1) \frac{q(x)}{p(x)} = \lim_{x \to \beta_1} (x - \beta_1) \frac{(x - \gamma_1)}{(x - \beta_1)} = \lim_{x \to \beta_1} (x - \gamma_1) = \beta_1 - \gamma_1 \,,$$

then $\beta_1 \leq \gamma_1$, and so $\{\gamma_1\}$ interlaces on $\{\beta_1\}$, as claimed. Now assume this direction of the proof holds for $M' = 1, \ldots, M-1$, and let $\{\beta_m\}_{m=1}^M$ and $\{\gamma_m\}_{m=1}^M$ be real, nonincreasing and have the property that

$$\lim_{x \to \beta_m} (x - \beta_m) \frac{q(x)}{p(x)} \leq 0 \qquad \forall m = 1, \ldots, M, \tag{3.15}$$

where $p(x)$ and $q(x)$ are defined as in the statement of the result. We will show that $\{\gamma_m\}_{m=1}^M$ interlaces on $\{\beta_m\}_{m=1}^M$.

To do this, we consider two cases. The first case is when $\{\beta_m\}_{m=1}^M$ and $\{\gamma_m\}_{m=1}^M$ have no common members, that is, $\beta_m \neq \gamma_{m'}$ for all $m, m' = 1, \ldots, M$. In this case, note that if $\beta_m = \beta_{m'}$ for some $m \neq m'$ then the corresponding limit in (3.15) would diverge, contradicting our implicit assumption that these limits exist and are nonpositive. As such, in this case the values of $\{\beta_m\}_{m=1}^M$ are necessarily distinct, at which point (3.15) for a given m becomes:

$$\begin{aligned} 0 &\geq \lim_{x \to \beta_m} (x - \beta_m) \prod_{m'=1}^{M} \frac{(x - \gamma_{m'})}{(x - \beta_{m'})} \\ &= \frac{\prod_{m'=1}^{M} (\beta_m - \gamma_{m'})}{\prod_{\substack{m'=1 \\ m' \neq m}}^{M} (\beta_m - \beta_{m'})} \\ &= \frac{\prod_{m'=1}^{M} (\beta_m - \gamma_{m'})}{\prod_{m'=1}^{m-1} (\beta_m - \beta_{m'}) \prod_{m'=m+1}^{M} (\beta_m - \beta_{m'})} . \end{aligned} \tag{3.16}$$

Moreover, since $\beta_m \neq \gamma_{m'}$ for all m, m', then the limit in (3.16) is nonzero. As the sign of the denominator on the right-hand side of (3.16) is $(-1)^{m-1}$, the sign of the corresponding

numerator is

$$(-1)^m = \operatorname{sgn}\Big(\prod_{m'=1}^{M}(\beta_m - \gamma_{m'})\Big) = \operatorname{sgn}(q(\beta_m)) \qquad \forall m = 1, \dots, M.$$

Thus, for any $m = 2, \dots, M$, $q(x)$ changes sign over $[\beta_m, \beta_{m-1}]$, implying by the Intermediate Value Theorem that at least one of the roots $\{\gamma_m\}_{m=1}^M$ of $q(x)$ lies in (β_m, β_{m-1}). Moreover, since $q(x)$ is monic, we have $\lim_{x\to\infty} q(x) = \infty$; coupled with the fact that $q(\beta_1) < 0$, this implies that at least one root of $q(x)$ lies in (β_1, ∞). Thus, each of the M disjoint subintervals of $(\beta_1, \infty) \cup [\cup_{m=2}^M (\beta_m, \beta_{m-1})]$ contains at least one of the M roots of $q(x)$. This is only possible if each of these subintervals contains exactly one of these roots. Moreover, since $\{\gamma_m\}_{m=1}^M$ is nonincreasing, this implies $\beta_1 < \gamma_1$ and $\beta_m < \gamma_m < \beta_{m-1}$ for all $m = 2, \dots, M$, meaning that $\{\gamma_m\}_{m=1}^M$ indeed interlaces on $\{\beta_m\}_{m=1}^M$.

We are thus left to consider the remaining case where $\{\beta_m\}_{m=1}^M$ and $\{\gamma_m\}_{m=1}^M$ share at least one common member. Fix λ such that $\beta_m = \lambda = \gamma_{m'}$ for at least one pair $m, m' = 1, \dots, M$. Let $m_p = \min\{m : \beta_m = \lambda\}$ and $m_q = \min\{m : \gamma_m = \lambda\}$. Let $P(x)$ and $Q(x)$ be $(M-1)$-degree polynomials such that $p(x) = (x - \lambda)P(x)$ and $q(x) = (x - \lambda)Q(x)$. Here, our assumption (3.15) implies

$$0 \ge \lim_{x\to\beta_m}(x - \beta_m)\frac{q(x)}{p(x)} = \lim_{x\to\beta_m}(x - \beta_m)\frac{(x-\lambda)Q(x)}{(x-\lambda)P(x)} = \lim_{x\to\beta_m}(x - \beta_m)\frac{Q(x)}{P(x)} \tag{3.17}$$

for all $m = 1, \dots, M$. Since $P(x)$ and $Q(x)$ satisfy (3.17) and have degree $M - 1$, our inductive hypothesis gives that the roots $\{\gamma_m\}_{m\neq m_q}$ of $Q(x)$ interlace on the roots $\{\beta_m\}_{m\neq m_p}$ of $P(x)$.

We claim that m_q is necessarily either m_p or $m_p + 1$, that is, $m_p \le m_q \le m_p + 1$. We first show that $m_p \le m_q$, a fact which trivially holds for $m_p = 1$. For $m_p > 1$, the fact that $\{\beta_m\}_{m\neq m_p} \sqsubseteq \{\gamma_m\}_{m\neq m_q}$ implies that the value of the $(m_p - 1)$th member of $\{\gamma_m\}_{m\neq m_q}$ is at least that of the $(m_p - 1)$th member of $\{\beta_m\}_{m\neq m_p}$. That is, the $(m_p - 1)$th member of $\{\gamma_m\}_{m\neq m_q}$ is at least $\beta_{m_p-1} > \lambda$, meaning $m_p - 1 \le m_q - 1$ and so $m_p \le m_q$, as claimed. We similarly prove that $m_q \le m_p + 1$, a fact which trivially holds for $m_p = M$. For $m_p < M$, interlacing implies

that the m_pth member of $\{\beta_m\}_{m\neq m_p}$ is at least the (m_p+1)th member of $\{\gamma_m\}_{m\neq m_q}$. That is, the (m_p+1)th member of $\{\gamma_m\}_{m\neq m_q}$ is at most $\beta_{m_p+1}\leq\lambda$ and so $m_p+1\geq m_q$, as claimed.

Now, in the case that $m_q=m_p$, the fact that $\{\beta_m\}_{m\neq m_p}\sqsubseteq\{\gamma_m\}_{m\neq m_q}$ implies that

$$\beta_M\leq\gamma_M\leq\cdots\leq\beta_{m_p+1}\leq\gamma_{m_p+1}\leq\beta_{m_p-1}\leq\gamma_{m_p-1}\leq\cdots\leq\beta_1\leq\gamma_1. \tag{3.18}$$

Since in this case $\gamma_{m_p+1}=\gamma_{m_q+1}\leq\lambda<\beta_{m_p-1}$, the terms $\beta_{m_p}=\lambda$ and $\gamma_{m_p}=\gamma_{m_q}=\lambda$ can be inserted into (3.18):

$$\beta_M\leq\gamma_M\leq\cdots\leq\beta_{m_p+1}\leq\gamma_{m_p+1}\leq\lambda\leq\lambda\leq\beta_{m_p-1}\leq\gamma_{m_p-1}\leq\cdots\leq\beta_1\leq\gamma_1,$$

and so $\{\beta_m\}_{m=1}^M\sqsubseteq\{\gamma_m\}_{m=1}$. In the remaining case where $m_q=m_p+1$, having $\{\beta_m\}_{m\neq m_p}\sqsubseteq\{\gamma_m\}_{m\neq m_q}$ means that

$$\beta_M\leq\gamma_M\leq\cdots\leq\beta_{m_p+2}\leq\gamma_{m_p+2}\leq\beta_{m_p+1}\leq\gamma_{m_p}\leq\beta_{m_p-1}\leq\gamma_{m_p-1}\leq\cdots\leq\beta_1\leq\gamma_1. \tag{3.19}$$

Since in this case $\beta_{m_p+1}\leq\lambda<\gamma_{m_q-1}=\gamma_{m_p}$, the terms $\gamma_{m_p+1}=\gamma_{m_q}=\lambda$ and $\beta_{m_p}=\lambda$ can be inserted into (3.19):

$$\beta_M\leq\gamma_M\leq\cdots\leq\gamma_{m_p+2}\leq\beta_{m_p+1}\leq\lambda\leq\lambda\leq\gamma_{m_p}\leq\beta_{m_p-1}\leq\cdots\leq\beta_1\leq\gamma_1$$

and so $\{\beta_m\}_{m=1}^M\sqsubseteq\{\gamma_m\}_{m=1}^M$ in this case as well. □

Lemma 6. *If $\{\beta_m\}_{m=1}^M$, $\{\gamma_m\}_{m=1}^M$, and $\{\delta_m\}_{m=1}^M$ are real and nonincreasing and*

$$\lim_{x\to\beta_m}(x-\beta_m)\frac{q(x)}{p(x)}=\lim_{x\to\beta_m}(x-\beta_m)\frac{r(x)}{p(x)}\qquad\forall m=1,\ldots,M,$$

where $p(x)=\prod_{m=1}^M(x-\beta_m)$, $q(x)=\prod_{m=1}^M(x-\gamma_m)$ and $r(x)=\prod_{m=1}^M(x-\delta_m)$, then $q(x)=r(x)$.

Proof. Fix any $m=1,\ldots,M$, and let L be the multiplicity of β_m as a root of $p(x)$. Since

$$\lim_{x\to\beta_m}(x-\beta_m)\frac{q(x)}{p(x)}=\lim_{x\to\beta_m}(x-\beta_m)\frac{r(x)}{p(x)}\,, \tag{3.20}$$

where each of these two limits is assumed to exist, then the multiplicities of β_m as a roots of $q(x)$ and $r(x)$ are both at least $L-1$. As such, evaluating lth derivatives at β_m gives

$q^{(l)}(\beta_m) = 0 = r^{(l)}(\beta_m)$ for all $l = 0, \ldots, L-2$. Meanwhile, for $l = L-1$, l'Hôpital's Rule gives

$$\begin{aligned}\lim_{x\to\beta_m}(x-\beta_m)\frac{q(x)}{p(x)} &= \lim_{x\to\beta_m}\frac{q(x)}{(x-\beta_m)^{L-1}}\frac{(x-\beta_m)^L}{p(x)}\\ &= \frac{q^{(L-1)}(\beta_m)}{(L-1)!}\frac{L!}{p^{(L)}(\beta_m)}\\ &= \frac{Lq^{(L-1)}(\beta_m)}{p^{(L)}(\beta_m)}.\end{aligned} \tag{3.21}$$

Deriving a similar expression for $r(x)$ and substituting both it and (3.21) into (3.20) yields $q^{(L-1)}(\beta_m) = r^{(L-1)}(\beta_m)$. As such, $q^{(l)}(\beta_m) = r^{(l)}(\beta_m)$ for all $l = 0, \ldots, L-1$. As this argument holds at every distinct β_m, we see that $q(x) - r(x)$ has M roots, counting multiplicity. But since $q(x)$ and $r(x)$ are both monic, $q(x)-r(x)$ has degree at most $M-1$ and so $q(x)-r(x) \equiv 0$, as claimed. □

With Theorem 4 and Lemmas 3, 5 and 6 in hand, we are ready to prove the main result of this section.

Proof of Theorem 2. ($\Rightarrow$) Let $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ be arbitrary nonnegative nonincreasing sequences, and let $F = \{f_n\}_{n=1}^N$ be any sequence of vectors such that the spectrum of FF^* is $\{\lambda_m\}_{m=1}^M$ and $\|f_n\|^2 = \mu_n$ for all $n = 1, \ldots, N$. We claim that this particular F can be constructed by following Steps A and B.

In particular, consider the sequence of sequences $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ defined by letting $\{\lambda_{n;m}\}_{m=1}^M$ be the spectrum of the frame operator $F_nF_n^*$ of the sequence $F_n = \{f_{n'}\}_{n'=1}^n$ for all $n = 1, \ldots, N$ and letting $\lambda_{0;m} = 0$ for all m. We claim that $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ satisfies Definition 1 and therefore is a valid sequence of outer eigensteps. Note conditions (i) and (ii) of Definition 1 are immediately satisfied. To see that $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ satisfies (iii), consider the polynomials $p_n(x)$ defined by (3.6) for all $n = 1, \ldots, N$. In the special case where $n = 1$, the desired property (iii) that $\{0\}_{m=1}^M \sqsubseteq \{\lambda_{1;m}\}_{m=1}^M$ follows from the fact that the spectrum $\{\lambda_{1;m}\}_{m=1}^M$ of the scaled rank-one projection $F_1F_1^* = f_1f_1^*$ is the value $\|f_1\|^2 = \mu_1$

along with $M-1$ repetitions of 0, the eigenspaces being the span of f_1 and its orthogonal complement, respectively. Meanwhile if $n=2,\dots,N$, Theorem 4 gives that

$$\lim_{x\to\lambda_{n-1;m}}(x-\lambda_{n-1;m})\frac{p_n(x)}{p_{n-1}(x)} = -\|P_{n-1;\lambda_{n-1;m}}f_n\|^2 \leq 0 \qquad \forall m=1,\dots,M,$$

implying by Lemma 5 that $\{\lambda_{n-1;m}\}_{m=1}^M \sqsubseteq \{\lambda_{n;m}\}_{m=1}^M$ as claimed. Finally, (iv) holds since for any $n=1,\dots,N$ we have

$$\sum_{m=1}^M \lambda_{n;m} = \mathrm{Tr}(F_nF_n^*) = \mathrm{Tr}(F_n^*F_n) = \sum_{n'=1}^n \|f_{n'}\|^2 = \sum_{n'=1}^n \mu_{n'}.$$

Having shown that these particular values of $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ can indeed be chosen in Step A, we next show that our particular F can be constructed according to Step B. As the method of Step B is iterative, we use induction to prove that it can yield F. Indeed, the only restriction that Step B places on f_1 is that $\|f_1\|^2=\mu_1$, something our particular f_1 satisfies by assumption. Now assume that for any $n=1,\dots,N-1$ we have already correctly produced $\{f_{n'}\}_{n'=1}^n$ by following the method of Step B; we show that we can produce the correct f_{n+1} by continuing to follow Step B. To be clear, each iteration of Step B does not produce a unique vector, but rather presents a family of f_{n+1}'s to choose from, and we show that our particular choice of f_{n+1} lies in this family. Specifically, our choice of f_{n+1} must satisfy (3.7) for any choice of $\lambda\in\{\lambda_{n;m}\}_{m=1}^M$; the fact that it indeed does so follows immediately from Theorem 4. To summarize, we have shown that by making appropriate choices, we can indeed produce our particular F by following Steps A and B, concluding this direction of the proof.

($\Leftarrow$) Now assume that a sequence of vectors $F=\{f_n\}_{n=1}^N$ has been produced according to Steps A and B. To be precise, letting $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ be the sequence of outer eigensteps chosen in Step A, we claim that any $F=\{f_n\}_{n=1}^N$ constructed according to Step B has the property that the spectrum of the frame operator $F_nF_n^*$ of $F_n=\{f_{n'}\}_{n'=1}^n$ is $\{\lambda_{n;m}\}_{m=1}^M$ for all $n=1,\dots,N$. Note that by Lemma 3, proving this claim will yield our stated result that the spectrum of FF^* is $\{\lambda_m\}_{m=1}^M$ and that $\|f_n\|^2=\mu_n$ for all $n=1,\dots,N$. As the method of Step B

is iterative, we prove this claim by induction. Step B begins by taking any f_1 such that $\|f_1\|^2 = \mu_1$. As noted above in the proof of the other direction, the spectrum of $F_1F_1^* = f_1f_1^*$ is the value μ_1 along with $M-1$ repetitions of 0. As claimed, these values match those of $\{\lambda_{1;m}\}_{m=1}^M$; to see this, note that Definition 1.(i) and (iii) give $\{0\}_{m=1}^M = \{\lambda_{0;m}\}_{m=1}^M \sqsubseteq \{\lambda_{1;m}\}_{m=1}^M$ and so $\lambda_{1;m} = 0$ for all $m = 2, \ldots, M$, at which point Definition 1.(iv) implies $\lambda_{1,1} = \mu_1$.

Now assume that for any $n = 1, \ldots, N-1$, the Step B process has already produced $F_n = \{f_{n'}\}_{n'=1}^n$ such that the spectrum of $F_nF_n^*$ is $\{\lambda_{n;m}\}_{m=1}^M$. We show that by following Step B, we produce an f_{n+1} such that $F_{n+1} = \{f_{n'}\}_{n'=1}^{n+1}$ has the property that $\{\lambda_{n+1;m}\}_{m=1}^M$ is the spectrum of $F_{n+1}F_{n+1}^*$. To do this, consider the polynomials $p_n(x)$ and $p_{n+1}(x)$ defined by (3.6) and pick any f_{n+1} that satisfies (3.7), namely

$$\lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{p_{n+1}(x)}{p_n(x)} = -\|P_{n;\lambda_{n;m}}f_{n+1}\|^2 \qquad \forall m = 1, \ldots, M. \tag{3.22}$$

Letting $\{\hat{\lambda}_{n+1;m}\}_{m=1}^M$ denote the spectrum of $F_{n+1}F_{n+1}^*$, our goal is to show that $\{\hat{\lambda}_{n+1;m}\}_{m=1}^M = \{\lambda_{n+1;m}\}_{m=1}^M$. Equivalently, our goal is to show that $p_{n+1}(x) = \hat{p}_{n+1}(x)$ where $\hat{p}_{n+1}(x)$ is the polynomial

$$\hat{p}_{n+1}(x) := \prod_{m=1}^M (x-\hat{\lambda}_{n+1;m}).$$

Since $p_n(x)$ and $\hat{p}_{n+1}(x)$ are the characteristic polynomials of $F_nF_n^*$ and $F_{n+1}F_{n+1}^*$, respectively, Theorem 4 gives:

$$\lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{\hat{p}_{n+1}(x)}{p_n(x)} = -\|P_{n;\lambda_{n;m}}f_{n+1}\|^2 \qquad \forall m = 1, \ldots, M. \tag{3.23}$$

Comparing (3.22) and (3.23) gives:

$$\lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{p_{n+1}(x)}{p_n(x)} = \lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{\hat{p}_{n+1}(x)}{p_n(x)} \qquad \forall m = 1, \ldots, M,$$

implying by Lemma 6 that $p_{n+1}(x) = \hat{p}_{n+1}(x)$, as desired. □

3.2 Constructing frame elements from eigensteps

In the previous section, we proved that Theorem 2 provides a two-step process for constructing any and all sequences of vectors $F = \{f_n\}_{n=1}^N$ in $\mathbb{H}_M$ whose frame operator

possesses a given spectrum $\{\lambda_m\}_{m=1}^M$ and whose vectors have given lengths $\{\mu_n\}_{n=1}^N$. In this section, we focus on improving Step B, namely how we can explicitly construct any and all sequences of vectors whose partial-frame-operator spectra match the outer eigensteps chosen in Step A. While the algorithm below appears very technical, it can nonetheless be performed by hand or numerically using programs such as MATLAB. MATLAB code for implementing this algorithm has been included in the appendix.

Theorem 7. *For any nonnegative nonincreasing sequences* $\{\lambda_m\}_{m=1}^M$ *and* $\{\mu_n\}_{n=1}^N$*, every sequence of vectors* $F = \{f_n\}_{n=1}^N$ *in* $\mathbb{H}_M$ *whose frame operator* FF^* *has spectrum* $\{\lambda_m\}_{m=1}^M$ *and which satisfies* $\|f_n\|^2 = \mu_n$ *for all* n *can be constructed by the following algorithm:*

A. Pick outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ *as in Definition 1.*

B. Let U_1 *be any unitary matrix,* $U_1 = \{u_{1;m}\}_{m=1}^M$*, and let* $f_1 = \sqrt{\mu_1}u_{1;1}$*. For each* $n = 1, \dots, N-1$*:*

B.1 Let V_n *be an* $M \times M$ *block-diagonal unitary matrix whose blocks correspond to the distinct values of* $\{\lambda_{n;m}\}_{m=1}^M$ *with the size of each block being the multiplicity of the corresponding eigenvalue.*

B.2 Identify terms which are common to both $\{\lambda_{n;m}\}_{m=1}^M$ *and* $\{\lambda_{n+1;m}\}_{m=1}^M$*. Specifically:*

- *Let* $\mathcal{I}_n \subseteq \{1, \dots, M\}$ *consist of those indices* m *such that* $\lambda_{n;m} < \lambda_{n;m'}$ *for all* $m' < m$ *and such that the multiplicity of* $\lambda_{n;m}$ *as a value in* $\{\lambda_{n;m'}\}_{m'=1}^M$ *exceeds its multiplicity as a value in* $\{\lambda_{n+1;m'}\}_{m'=1}^M$*.*
- *Let* $\mathcal{J}_n \subseteq \{1, \dots, M\}$ *consist of those indices* m *such that* $\lambda_{n+1;m} < \lambda_{n+1;m'}$ *for all* $m' < m$ *and such that the multiplicity of* $\lambda_{n;m}$ *as a value in* $\{\lambda_{n+1;m'}\}_{m'=1}^M$ *exceeds its multiplicity as a value in* $\{\lambda_{n;m'}\}_{m'=1}^M$*.*

The sets $\mathcal{I}_n$ *and* $\mathcal{J}_n$ *have equal cardinality, which we denote* R_n*. Next:*

- *Let $\pi_{\mathcal{I}_n}$ be the unique permutation on $\{1,\dots,M\}$ that is increasing on both $\mathcal{I}_n$ and $\mathcal{I}_n^{\mathrm{c}}$ and such that $\pi_{\mathcal{I}_n}(m)\in\{1,\dots,R_n\}$ for all $m\in\mathcal{I}_n$. Let $\Pi_{\mathcal{I}_n}$ be the associated permutation matrix $\Pi_{\mathcal{I}_n}\delta_m=\delta_{\pi_{\mathcal{I}_n}(m)}$.*
- *Let $\pi_{\mathcal{J}_n}$ be the unique permutation on $\{1,\dots,M\}$ that is increasing on both $\mathcal{J}_n$ and $\mathcal{J}_n^{\mathrm{c}}$ and such that $\pi_{\mathcal{J}_n}(m)\in\{1,\dots,R_n\}$ for all $m\in\mathcal{J}_n$. Let $\Pi_{\mathcal{J}_n}$ be the associated permutation matrix $\Pi_{\mathcal{J}_n}\delta_m=\delta_{\pi_{\mathcal{J}_n}(m)}$.*

B.3 Let v_n, w_n be the $R_n\times 1$ vectors whose entries are

$$v_n(\pi_{\mathcal{I}_n}(m))=\left[-\frac{\prod\limits_{m''\in\mathcal{J}_n}(\lambda_{n;m}-\lambda_{n+1,m''})}{\prod\limits_{\substack{m''\in\mathcal{I}_n\\ m''\neq m}}(\lambda_{n;m}-\lambda_{n;m''})}\right]^{\frac{1}{2}}\qquad\forall m\in\mathcal{I}_n$$

$$w_n(\pi_{\mathcal{J}_n}(m'))=\left[\frac{\prod\limits_{m''\in\mathcal{I}_n}(\lambda_{n+1;m'}-\lambda_{n;m''})}{\prod\limits_{\substack{m''\in\mathcal{J}_n\\ m''\neq m'}}(\lambda_{n+1;m'}-\lambda_{n+1;m''})}\right]^{\frac{1}{2}}\qquad\forall m'\in\mathcal{J}_n.$$

B.4 $f_{n+1}=U_nV_n\Pi_{\mathcal{I}_n}^{\mathrm{T}}\begin{bmatrix}v_n\\0\end{bmatrix}$, where the $M\times 1$ vector $\begin{bmatrix}v_n\\0\end{bmatrix}$ is v_n padded with $M-R_n$ zeros.

B.5 $U_{n+1}=U_nV_n\Pi_{\mathcal{I}_n}^{\mathrm{T}}\begin{bmatrix}W_n&0\\0&\mathrm{I}\end{bmatrix}\Pi_{\mathcal{J}_n}$ where W_n is the $R_n\times R_n$ matrix whose entries are:

$$W_n(\pi_{\mathcal{I}_n}(m),\pi_{\mathcal{J}_n}(m'))=\frac{1}{\lambda_{n+1;m'}-\lambda_{n;m}}v_n(\pi_{\mathcal{I}_n}(m))w_n(\pi_{\mathcal{J}_n}(m')).$$

Conversely, any F constructed by this process has $\{\lambda_m\}_{m=1}^M$ as the spectrum of FF^* and $\|f_n\|^2=\mu_n$ for all n.

Moreover, for any F constructed in this manner and any $n=1,\dots,N$, the spectrum of the frame operator $F_nF_n^*$ arising from the partial sequence $F_n=\{f_{n'}\}_{n'=1}^n$ is $\{\lambda_{n;m}\}_{m=1}^M$, and the columns of U_n form a corresponding orthonormal eigenbasis for $F_nF_n^*$.

Before proving Theorem 7, we give an example of its implementation, with the hope of conveying the simplicity of the underlying idea, and better explaining the heavy notation used in the statement of the result.

Example 8. We now use Theorem 7 to construct UNTFs consisting of 5 vectors in $\mathbb{R}^3$. Here, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{5}{3}$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 1$. By Step A, our first task is to pick a sequence of outer eigensteps consistent with Definition 1, that is, pick $\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\}$, $\{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\}$, $\{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\}$ and $\{\lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}\}$ that satisfy the interlacing conditions:

$$\begin{aligned}
\{0,0,0\} &\sqsubseteq \{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\}\\
\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} &\sqsubseteq \{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\}\\
\{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} &\sqsubseteq \{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\}\\
\{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} &\sqsubseteq \{\lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}\}\\
\{\lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}\} &\sqsubseteq \{\tfrac{5}{3}, \tfrac{5}{3}, \tfrac{5}{3}\},
\end{aligned} \tag{3.24}$$

as well as the trace conditions:

$$\begin{aligned}
\lambda_{1;1} + \lambda_{1;2} + \lambda_{1;3} = 1, \quad \lambda_{2;1} + \lambda_{2;2} + \lambda_{2;3} = 2,\\
\lambda_{3;1} + \lambda_{3;2} + \lambda_{3;3} = 3, \quad \lambda_{4;1} + \lambda_{4;2} + \lambda_{4;3} = 4.
\end{aligned} \tag{3.25}$$

Writing these desired spectra in a table:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	?	?	?	?	$\frac{5}{3}$
$\lambda_{n;2}$	0	?	?	?	?	$\frac{5}{3}$
$\lambda_{n;1}$	0	?	?	?	?	$\frac{5}{3}$

the trace condition (3.25) means that the sum of the values in the nth column is $\sum_{n'=1}^{n} \mu_{n'} = n$, while the interlacing condition (3.24) means that any value $\lambda_{n;m}$ is at least the neighbor to the upper right $\lambda_{n+1;m+1}$ and no more than its neighbor to the right $\lambda_{n+1;m}$. In particular,

for $n = 1$, we necessarily have $0 = \lambda_{0;2} \le \lambda_{1;2} \le \lambda_{0;1} = 0$ and $0 = \lambda_{0;3} \le \lambda_{1;3} \le \lambda_{0;2} = 0$ implying that $\lambda_{1;2} = \lambda_{1;3} = 0$. Similarly, for $n = 4$, interlacing requires that $\frac{5}{3} = \lambda_{5;2} \le \lambda_{4;1} \le \lambda_{5;1} = \frac{5}{3}$ and $\frac{5}{3} = \lambda_{5;3} \le \lambda_{4;2} \le \lambda_{5;2} = \frac{5}{3}$ implying that $\lambda_{4;1} = \lambda_{4;2} = \frac{5}{3}$. That is, we necessarily have:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	?	?	?	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	?	?	?	$\frac{5}{3}$	$\frac{5}{3}$

Applying this same idea again for $n = 2$ and $n = 3$ gives $0 = \lambda_{1;3} \le \lambda_{2;3} \le \lambda_{1;2} = 0$ and $\frac{5}{3} = \lambda_{4;2} \le \lambda_{3;1} \le \lambda_{4;1} = \frac{5}{3}$, and so we also necessarily have that $\lambda_{2;3} = 0$, and $\lambda_{3;1} = \frac{5}{3}$:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	0	?	?	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

Moreover, the trace condition (3.25) at $n = 1$ gives $1 = \lambda_{1;1} + \lambda_{1;2} + \lambda_{1;3} = \lambda_{1;1} + 0 + 0$ and so $\lambda_{1;1} = 1$. Similarly, the trace condition at $n = 4$ gives $4 = \lambda_{4;1} + \lambda_{4;2} + \lambda_{4;3} = \frac{5}{3} + \frac{5}{3} + \lambda_{4;3}$ and so $\lambda_{4;3} = \frac{2}{3}$:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	0	?	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	1	?	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

The remaining entries are not fixed. In particular, we let $\lambda_{3;3}$ be some variable x and note that by the trace condition, $3 = \lambda_{3;1} + \lambda_{3;2} + \lambda_{3;3} = x + \lambda_{3;2} + \frac{5}{3}$ and so $\lambda_{3;2} = \frac{4}{3} - x$. Similarly

letting $\lambda_{2;2} = y$ gives $\lambda_{2;1} = 2 - y$:

$$\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \lambda_{n;3} & 0 & 0 & 0 & x & \frac{2}{3} & \frac{5}{3} \\ \lambda_{n;2} & 0 & 0 & y & \frac{4}{3} - x & \frac{5}{3} & \frac{5}{3} \\ \lambda_{n;1} & 0 & 1 & 2 - y & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} \end{array} \tag{3.26}$$

We take care to note that x and y in (3.26) are not arbitrary, but instead must be chosen so that the interlacing relations (3.26) are satisfied. In particular, we have:

$$\begin{aligned} \{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} \sqsubseteq \{\lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}\} \quad &\Longleftrightarrow \quad x \leq \tfrac{2}{3} \leq \tfrac{4}{3} - x \leq \tfrac{5}{3}, \\ \{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} \sqsubseteq \{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} \quad &\Longleftrightarrow \quad 0 \leq x \leq y \leq \tfrac{4}{3} - x \leq 2 - y \leq \tfrac{5}{3}, \\ \{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} \sqsubseteq \{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} \quad &\Longleftrightarrow \quad 0 \leq y \leq 1 \leq 2 - y. \end{aligned} \tag{3.27}$$

By plotting each of the 11 inequalities of (3.27) as a half-plane (Figure 3.1(a)), we obtain a 5-sided convex set (Figure 3.1(b)) of all (x, y) such that (3.26) is a valid sequence of outer eigensteps. Specifically, this set is the convex hull of $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{2}{3})$, $(\frac{1}{3}, 1)$ and $(0, \frac{2}{3})$. We note that though this analysis is straightforward in this case, it does not easily generalize to other cases in which M and N are large.

To complete Step A of Theorem 7, we pick any particular (x, y) from the set depicted in Figure 3.1(b). For example, if we pick $(x, y) = (0, \frac{1}{3})$ then (3.26) becomes:

$$\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \lambda_{n;3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{5}{3} \\ \lambda_{n;2} & 0 & 0 & \frac{1}{3} & \frac{4}{3} & \frac{5}{3} & \frac{5}{3} \\ \lambda_{n;1} & 0 & 1 & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} \end{array} \tag{3.28}$$

We now perform Step B of Theorem 7 for this particular choice of outer eigensteps. First, we must choose a unitary matrix U_1. Considering the equation for U_{n+1} along with the fact

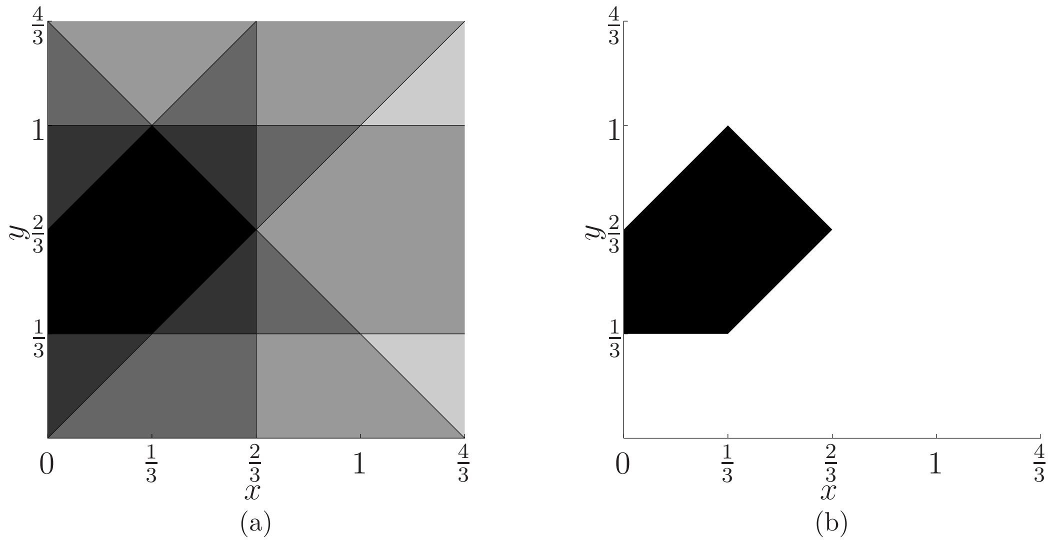

Figure 3.1: Pairs of parameters (x, y) that generate a valid sequence of outer eigensteps when substituted into (3.26). To be precise, in order to satisfy the interlacing requirements of Definition 1, x and y must be chosen so as to satisfy the 11 pairwise inequalities summarized in (3.27). Each of these inequalities corresponds to a half-plane (a), and the set of (x, y) that satisfy all of them is given by their intersection (b). By Theorem 7, any corresponding sequence of outer eigensteps (3.26) generates a 3×5 UNTF and conversely, every 3×5 UNTF is generated in this way. As such, x and y may be viewed as the two essential parameters in the set of all such frames.

that the columns of U_N will form an eigenbasis for F, we see that our choice for U_1 merely rotates this eigenbasis, and hence the entire frame F, to our liking. We choose $U_1 = \mathrm{I}$ for the sake of simplicity. Thus,

$$f_1 = \sqrt{\mu_1} u_{1;1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We now iterate, performing Steps B.1 through B.5 for $n = 1$ to find f_2 and U_2, then performing Steps B.1 through B.5 for $n = 2$ to find f_3 and U_3, and so on. Throughout this process, the only remaining choices to be made appear in Step B.1. In particular, for $n = 1$ Step B.1 asks us to pick a block-diagonal unitary matrix V_1 whose blocks are sized according to the multiplicities of the eigenvalues $\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} = \{1, 0, 0\}$. That

is, V_1 consists of a 1×1 unitary block—a unimodular scalar—and a 2×2 unitary block. There are an infinite number of such V_1's, each leading to a distinct frame. For the sake of simplicity, we choose $V_1 = \mathrm{I}$. Having completed Step B.1 for $n = 1$, we turn to Step B.2, which requires us to consider the columns of (3.28) that correspond to $n = 1$ and $n = 2$:

$$\begin{array}{c|cc} n & 1 & 2 \\ \hline \lambda_{n;3} & 0 & 0 \\ \lambda_{n;2} & 0 & \frac{1}{3} \\ \lambda_{n;1} & 1 & \frac{5}{3} \end{array} \tag{3.29}$$

In particular, we compute a set of indices $\mathcal{I}_1 \subseteq \{1, 2, 3\}$ that contains the indices m of $\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} = \{1, 0, 0\}$ for which (i) the multiplicity of $\lambda_{1;m}$ as a value of $\{1, 0, 0\}$ exceeds its multiplicity as a value of $\{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} = \{\frac{5}{3}, \frac{1}{3}, 0\}$ and (ii) m corresponds to the first occurrence of $\lambda_{1;m}$ as a value of $\{1, 0, 0\}$; by these criteria, we find $\mathcal{I}_1 = \{1, 2\}$. Similarly $m \in \mathcal{J}_1$ if and only if m indicates the first occurrence of a value $\lambda_{2;m}$ whose multiplicity as a value of $\{\frac{5}{3}, \frac{1}{3}, 0\}$ exceeds its multiplicity as a value of $\{1, 0, 0\}$, and so $\mathcal{J}_1 = \{1, 2\}$. Equivalently, $\mathcal{I}_1$ and $\mathcal{J}_1$ can be obtained by canceling common terms from (3.29), working top to bottom; an explicit algorithm for doing so is given in Table 3.2 near the end of this chapter.

Continuing with Step B.2 for $n = 1$, we now find the unique permutation $\pi_{\mathcal{I}_1} : \{1, 2, 3\} \to \{1, 2, 3\}$ that is increasing on both $\mathcal{I}_1 = \{1, 2\}$ and its complement $\mathcal{I}_1^{\mathrm{c}} = \{3\}$ and takes $\mathcal{I}_1$ to the first $R_1 = |\mathcal{I}_1| = 2$ elements of $\{1, 2, 3\}$. In this particular instance, $\pi_{\mathcal{I}_1}$ happens to be the identity permutation, and so $\Pi_{\mathcal{I}_1} = \mathrm{I}$. Since $\mathcal{J}_1 = \{1, 2\} = \mathcal{I}_1$, we similarly have that $\pi_{\mathcal{J}_1}$ and $\Pi_{\mathcal{J}_1}$ are the identity permutation and matrix, respectively.

For the remaining steps, it is useful to isolate the terms in (3.29) that correspond to $\mathcal{I}_1$ and $\mathcal{J}_1$:

$$\begin{aligned} \beta_2 &= \lambda_{1;2} = 0, \quad \gamma_2 = \lambda_{2;2} = \tfrac{1}{3}, \\ \beta_1 &= \lambda_{1;1} = 1, \quad \gamma_1 = \lambda_{2;1} = \tfrac{5}{3}. \end{aligned} \tag{3.30}$$

In particular, in Step B.3, we find the $R_1 \times 1 = 2 \times 1$ vector v_1 by computing quotients of products of differences of the values in (3.30):

$$[v_1(1)]^2 = -\frac{(\beta_1 - \gamma_1)(\beta_1 - \gamma_2)}{(\beta_1 - \beta_2)} = -\frac{(1 - \frac{5}{3})(1 - \frac{1}{3})}{(1 - 0)} = \tfrac{4}{9}, \tag{3.31}$$

$$[v_1(2)]^2 = -\frac{(\beta_2 - \gamma_1)(\beta_2 - \gamma_2)}{(\beta_2 - \beta_1)} = -\frac{(0 - \frac{5}{3})(0 - \frac{1}{3})}{(0 - 1)} = \tfrac{5}{9}, \tag{3.32}$$

yielding $v_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{\sqrt{5}}{3} \end{bmatrix}$. Similarly, we compute $w_1 = \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ according to the formulas:

$$[w_1(1)]^2 = \frac{(\gamma_1 - \beta_1)(\gamma_1 - \beta_2)}{(\gamma_1 - \gamma_2)} = \frac{(\frac{5}{3} - 1)(\frac{5}{3} - 0)}{(\frac{5}{3} - \frac{1}{3})} = \tfrac{5}{6}, \tag{3.33}$$

$$[w_1(2)]^2 = \frac{(\gamma_2 - \beta_1)(\gamma_2 - \beta_2)}{(\gamma_2 - \gamma_1)} = \frac{(\frac{1}{3} - 1)(\frac{1}{3} - 0)}{(\frac{1}{3} - \frac{5}{3})} = \tfrac{1}{6}. \tag{3.34}$$

Next, in Step B.4, we form our second frame element $f_2 = U_1 V_1 \Pi_{\mathcal{I}_1}^{\mathrm{T}} \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$:

$$f_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{\sqrt{5}}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{\sqrt{5}}{3} \\ 0 \end{bmatrix}.$$

As justified in the proof of Theorem 7, the resulting partial sequence of vectors

$$F_2 = \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} \\ 0 & 0 \end{bmatrix}$$

has a frame operator $F_2 F_2^*$ whose spectrum is $\{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} = \{\frac{5}{3}, \frac{1}{3}, 0\}$. Moreover, a corresponding orthonormal eigenbasis for $F_2 F_2^*$ is computed in Step B.5; here the first step is to compute the $R_1 \times R_1 = 2 \times 2$ matrix W_1 by computing a pointwise product of a certain 2×2 matrix with the outer product of v_1 with w_1:

$$W_1 = \begin{bmatrix} \frac{1}{\gamma_1 - \beta_1} & \frac{1}{\gamma_2 - \beta_1} \\ \frac{1}{\gamma_1 - \beta_2} & \frac{1}{\gamma_2 - \beta_2} \end{bmatrix} \odot \begin{bmatrix} v_1(1) \\ v_1(2) \end{bmatrix} \begin{bmatrix} w_1(1) & w_1(2) \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{5} & 3 \end{bmatrix} \odot \begin{bmatrix} \frac{2\sqrt{5}}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} \\ \frac{5}{3\sqrt{6}} & \frac{\sqrt{5}}{3\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}.$$

Note that W_1 is a real orthogonal matrix whose diagonal and subdiagonal entries are strictly positive and whose superdiagonal entries are strictly negative; one can easily verify that every W_n has this form. More significantly, the proof of Theorem 7 guarantees that the columns of

$$\begin{aligned}
U_2 &= U_1 V_1 \Pi_{\mathcal{I}_1}^{\mathrm{T}} \begin{bmatrix} W_1 & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

form an orthonormal eigenbasis of $F_2 F_2^*$. This completes the $n = 1$ iteration of Step B; we now repeat this process for $n = 2, 3, 4$. For $n = 2$, in Step B.1 we arbitrarily pick some 3×3 diagonal unitary matrix V_2. Note that if we wish our frame to be real, there are only $2^3 = 8$ such choices of V_2. For the sake of simplicity, we choose $V_2 = \mathrm{I}$ in this example. Continuing, Step B.2 involves canceling the common terms in

n	2	3
$\lambda_{n;3}$	0	0
$\lambda_{n;2}$	$\frac{1}{3}$	$\frac{4}{3}$
$\lambda_{n;1}$	$\frac{5}{3}$	$\frac{5}{3}$

to find $\mathcal{I}_2 = \mathcal{J}_2 = \{2\}$, and so

$$\Pi_{\mathcal{I}_2} = \Pi_{\mathcal{J}_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In Step B.3, we find that $v_2 = w_2 = \begin{bmatrix} 1 \end{bmatrix}$. Steps B.4 and B.5 then give that $F_3 = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ and U_3 are

$$F_3 = \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{\sqrt{6}} \\ 0 & 0 & 0 \end{bmatrix}, \qquad U_3 = \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The columns of U_3 form an orthonormal eigenbasis for the partial frame operator $F_3F_3^*$ with corresponding eigenvalues $\{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} = \{\frac{5}{3}, \frac{4}{3}, 0\}$. For the $n = 3$ iteration, we pick $V_3 = \mathrm{I}$ and cancel the common terms in

n	3	4
$\lambda_{n;3}$	0	$\frac{2}{3}$
$\lambda_{n;2}$	$\frac{4}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	$\frac{5}{3}$	$\frac{5}{3}$

to obtain $\mathcal{I}_3 = \{2, 3\}$ and $\mathcal{J}_3 = \{1, 3\}$, implying

$$\Pi_{\mathcal{I}_3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \Pi_{\mathcal{J}_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \begin{aligned} \beta_2 &= \lambda_{3;3} = 0, & \gamma_2 &= \lambda_{4;3} = \tfrac{2}{3}, \\ \beta_1 &= \lambda_{3;2} = \tfrac{4}{3}, & \gamma_1 &= \lambda_{4;1} = \tfrac{5}{3}. \end{aligned}$$

In Step B.3, we then compute the $R_3 \times 1 = 2 \times 1$ vectors v_3 and w_3 in a manner analogous to (3.31), (3.32), (3.33) and (3.34):

$$v_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}, \qquad w_3 = \begin{bmatrix} \frac{\sqrt{5}}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Note that in Step B.4, the role of permutation matrix $\Pi_{\mathcal{I}_3}^{\mathrm{T}}$ is that it maps the entries of v_3 onto the $\mathcal{I}_3$ indices, meaning that v_4 lies in the span of the corresponding eigenvectors

$\{u_{3;m}\}_{m\in\mathcal{I}_3}$:

$$f_4 = U_3 V_3 \Pi_{\mathcal{I}_3}^{\mathrm{T}} \begin{bmatrix} v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} \\ \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}.$$

In a similar fashion, the purpose of the permutation matrices in Step B.5 is to embed the entries of the 2×2 matrix W_3 into the $\mathcal{I}_3 = \{2, 3\}$ rows and $\mathcal{J}_3 = \{1, 3\}$ columns of a 3×3 matrix:

$$U_4 = U_3 V_3 \Pi_{\mathcal{I}_3}^{\mathrm{T}} \begin{bmatrix} W_3 & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_3}$$

$$= \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{5}}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{\sqrt{6}} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{\sqrt{6}} & -\frac{\sqrt{5}}{6} \\ \frac{1}{\sqrt{6}} & 0 & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}.$$

For the last iteration $n = 4$, we again choose $V_4 = \mathrm{I}$ in Step B.1. For Step B.2, note that since

n	4	5
$\lambda_{n;3}$	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;2}$	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	$\frac{5}{3}$	$\frac{5}{3}$

we have $\mathcal{I}_4 = \{3\}$ and $\mathcal{J}_4 = \{1\}$, implying

$$\Pi_{\mathcal{I}_4} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \Pi_{\mathcal{J}_4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Working through Steps B.3, B.4 and B.5 yields the UNTF:

$$F = F_5 = \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{6} & \frac{1}{6} \\ 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{6} & -\frac{\sqrt{5}}{6} \\ 0 & 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}, \qquad U_5 = \begin{bmatrix} \frac{1}{6} & -\frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{\sqrt{6}} \\ -\frac{\sqrt{5}}{6} & \frac{5}{6} & \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \end{bmatrix}. \tag{3.35}$$

We emphasize that the UNTF F given in (3.35) was based on the particular choice of outer eigensteps given in (3.28), which arose by choosing $(x, y) = (0, \frac{1}{3})$ in (3.26). Choosing other pairs (x, y) from the parameter set depicted in Figure 3.1(b) yields other UNTFs. Indeed, since the outer eigensteps of a given F are equal to those of UF for any unitary operator U, we have in fact that each distinct (x, y) yields a UNTF which is not unitarily equivalent to any of the others. For example, by following the algorithm of Theorem 7 and choosing $U_1 = \mathrm{I}$ and $V_n = \mathrm{I}$ in each iteration, we obtain the following four additional UNTFs, each

corresponding to a distinct corner point from Figure 3.1(b) of the parameter set:

$$F = \begin{bmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{\sqrt{5}}{3} & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \\ 0 & 0 & 1 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \qquad \text{for } (x,y) = (\tfrac{1}{3}, \tfrac{1}{3}),$$

$$F = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{8}}{3} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \end{bmatrix} \qquad \text{for } (x,y) = (\tfrac{2}{3}, \tfrac{2}{3}),$$

$$F = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \end{bmatrix} \qquad \text{for } (x,y) = (\tfrac{1}{3}, 1),$$

$$F = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{\sqrt{3}} & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{\sqrt{8}}{3} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ 0 & 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix} \qquad \text{for } (x,y) = (0, \tfrac{2}{3}).$$

Notice that, of the four UNTFs above, the second and fourth are actually the same up to a permutation of the frame elements. This is an artifact of our method of construction, namely, that our choices for outer eigensteps, U_1, and $\{V_n\}_{n=1}^{N-1}$ determine the *sequence* of frame elements. As such, we can recover all permutations of a given frame by modifying these choices.

We emphasize that these four UNTFs along with that of (3.35) are but five examples from the continuum of all such frames. Indeed, keeping x and y as variables in (3.26) and applying the algorithm of Theorem 7—again choosing $U_1 = \mathrm{I}$ and $V_n = \mathrm{I}$ in each iteration for the sake of simplicity—yields the frame elements given in Table 3.1. Here, we restrict (x, y) so as to not lie on the boundary of the parameter set of Figure 3.1(b). This restriction simplifies the analysis, as it prevents all unnecessary repetitions of values

in neighboring columns in (3.26). Table 3.1 gives an explicit parametrization for a two-dimensional manifold that lies within the set of all UNTFs consisting of five elements in three-dimensional space. By Theorem 7, this can be generalized so as to yield all such frames, provided we both (i) further consider (x, y) that lie on each of the five line segments that constitute the boundary of the parameter set and (ii) throughout generalize V_n to an arbitrary block-diagonal unitary matrix, where the sizes of the blocks are chosen in accordance with Step B.1.

Having discussed the utility of Theorem 7, we turn to its proof.

Proof of Theorem 7. ($\Leftarrow$) Let $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ be arbitrary nonnegative nonincreasing sequences and take an arbitrary sequence of outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ in accordance with Definition 1. Note here we do not assume that such a sequence of outer eigensteps actually exists for this particular choice of $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$; if one does not, then this direction of the result is vacuously true.

We claim that any $F = \{f_n\}_{n=1}^N$ constructed according to Step B has the property that for all $n = 1, \dots, N$, the spectrum of the frame operator $F_n F_n^*$ of $F_n = \{f_{n'}\}_{n'=1}^n$ is $\{\lambda_{n;m}\}_{m=1}^M$, and that the columns of U_n form an orthonormal eigenbasis for $F_n F_n^*$. Note that by Lemma 3, proving this claim will yield our stated result that the spectrum of FF^* is $\{\lambda_m\}_{m=1}^M$ and that $\|f_n\|^2 = \mu_n$ for all $n = 1, \dots, N$. Since Step B is an iterative algorithm, we prove this claim by induction on n. To be precise, Step B begins by letting $U_1 = \{u_{1;m}\}_{m=1}^M$ and $f_1 = \sqrt{\mu_1} u_{1;1}$. The columns of U_1 form an orthonormal eigenbasis for $F_1 F_1^*$ since U_1 is unitary by assumption and

$$F_1 F_1^* u_{1;m} = \langle u_{1;m}, f_1 \rangle f_1 = \langle u_{1;m}, \sqrt{\mu_1} u_{1;1} \rangle \sqrt{\mu_1} u_{1;1} = \mu_1 \langle u_{1;m}, u_{1;1} \rangle u_{1;1} = \begin{cases} \mu_1 u_{1;1} & m = 1, \\ 0 & m \neq 1, \end{cases}$$

for all $m = 1, \dots, M$. As such, the spectrum of $F_1 F_1^*$ consists of μ_1 and $M-1$ repetitions of 0. To see that this spectrum matches the values of $\{\lambda_{1;m}\}_{m=1}^M$, note that by Definition 1, we

Table 3.1: A continuum of UNTFs. To be precise, for each choice of (x, y) that lies in the interior of the parameter set depicted in Figure 3.1(b), these five elements form a UNTF for $\mathbb{R}^3$, meaning that its 3×5 synthesis matrix F has both unit norm columns and orthogonal rows of constant squared norm $\frac{5}{3}$. These frames were produced by applying the algorithm of Theorem 7 to the sequence of outer eigensteps given in (3.26), choosing $U_1 = \mathrm{I}$ and $V_n = \mathrm{I}$ for all n. These formulas give an explicit parametrization for a two-dimensional manifold that lies within the set of all 3×5 UNTFs. By Theorem 7, every such UNTF arises in this manner, with the understanding that (x, y) may indeed be chosen from the boundary of the parameter set and that the initial eigenbasis U_1 and the block-diagonal unitary matrices V_n are not necessarily the identity.

$$f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$f_2 = \begin{bmatrix} 1-y \\ \sqrt{y(2-y)} \\ 0 \end{bmatrix}$$

$$f_3 = \begin{bmatrix} \frac{\sqrt{(3y-1)(2+3x-3y)(2-x-y)}}{6\sqrt{1-y}} - \frac{\sqrt{(5-3y)(4-3x-3y)(y-x)}}{6\sqrt{1-y}} \\ \frac{\sqrt{y(3y-1)(2+3x-3y)(2-x-y)}}{6\sqrt{(1-y)(2-y)}} + \frac{\sqrt{(5-3y)(2-y)(4-3x-3y)(y-x)}}{6\sqrt{y(1-y)}} \\ \frac{\sqrt{5x(4-3x)}}{3\sqrt{y(2-y)}} \end{bmatrix}$$

$$f_4 = \begin{bmatrix} -\frac{\sqrt{(4-3x)(3y-1)(2-x-y)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)}} - \frac{\sqrt{(4-3x)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)(1-y)}} - \frac{\sqrt{x(3y-1)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} + \frac{\sqrt{x(5-3y)(2-x-y)(4-3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} \\ -\frac{\sqrt{(4-3x)y(3y-1)(2-x-y)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)(2-y)}} + \frac{\sqrt{(4-3x)(2-y)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)y(1-y)}} - \frac{\sqrt{xy(3y-1)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)(2-y)}} - \frac{\sqrt{x(2-y)(5-3y)(2-x-y)(4-3x-3y)}}{4\sqrt{3(2-3x)y(1-y)}} \\ \frac{\sqrt{5x(2+3x-3y)(4-3x-3y)}}{6\sqrt{(2-3x)y(2-y)}} + \frac{\sqrt{5(4-3x)(y-x)(2-x-y)}}{2\sqrt{3(2-3x)y(2-y)}} \end{bmatrix}$$

$$f_5 = \begin{bmatrix} \frac{\sqrt{(4-3x)(3y-1)(2-x-y)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)}} + \frac{\sqrt{(4-3x)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)(1-y)}} - \frac{\sqrt{x(3y-1)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} + \frac{\sqrt{x(5-3y)(2-x-y)(4-3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} \\ \frac{\sqrt{(4-3x)y(3y-1)(2-x-y)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)(2-y)}} - \frac{\sqrt{(4-3x)(2-y)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)y(1-y)}} - \frac{\sqrt{xy(3y-1)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)(2-y)}} - \frac{\sqrt{x(2-y)(5-3y)(2-x-y)(4-3x-3y)}}{4\sqrt{3(2-3x)y(1-y)}} \\ -\frac{\sqrt{5x(2+3x-3y)(4-3x-3y)}}{6\sqrt{(2-3x)y(2-y)}} + \frac{\sqrt{5(4-3x)(y-x)(2-x-y)}}{2\sqrt{3(2-3x)y(2-y)}} \end{bmatrix}$$

know $\{\lambda_{1;m}\}_{m=1}^M$ interlaces on the trivial sequence $\{\lambda_{0;m}\}_{m=1}^M = \{0\}_{m=1}^M$ in the sense of (3.3), implying $\lambda_{1;m} = 0$ for all $m \geq 2$; this in hand, note this definition further gives that $\lambda_{1;1} = \sum_{m=1}^M \lambda_{1;m} = \mu_1$. Thus, our claim indeed holds for $n = 1$.

We now proceed by induction, assuming that for any given $n = 1, \dots, N-1$ the process of Step B has produced $F_n = \{f_{n'}\}_{n'=1}^n$ such that the spectrum of $F_n F_n^*$ is $\{\lambda_{n;m}\}_{m=1}^M$ and that the columns of U_n form an orthonormal eigenbasis for $F_n F_n^*$. In particular, we have $F_n F_n^* U_n = U_n D_n$ where D_n is the diagonal matrix whose diagonal entries are $\{\lambda_{n;m}\}_{m=1}^M$.

Defining D_{n+1} analogously from $\{\lambda_{n+1;m}\}_{m=1}^{M}$, we show that constructing f_{n+1} and U_{n+1} according to Step B implies $F_{n+1}F_{n+1}^{*}U_{n+1} = U_{n+1}D_{n+1}$ where U_{n+1} is unitary; doing such proves our claim.

To do so, pick any unitary matrix V_n according to Step B.1. To be precise, let K_n denote the number of distinct values in $\{\lambda_{n;m}\}_{m=1}^{M}$, and for any $k = 1, \dots, K_n$, let $L_{n;k}$ denote the multiplicity of the kth value. We write the index m as an increasing function of k and l, that is, we write $\{\lambda_{n;m}\}_{m=1}^{M}$ as $\{\lambda_{n;m(k,l)}\}_{k=1}^{K_n}{}_{l=1}^{L_{n;k}}$ where $m(k,l) < m(k',l')$ if $k < k'$ or if $k = k'$ and $l < l'$. We let V_n be an $M \times M$ block-diagonal unitary matrix consisting of K diagonal blocks, where for any $k = 1, \dots, K$, the kth block is an $L_{n;k} \times L_{n;k}$ unitary matrix. In the extreme case where all the values of $\{\lambda_{n;m}\}_{m=1}^{M}$ are distinct, we have that V_n is a diagonal unitary matrix, meaning it is a diagonal matrix whose diagonal entries are unimodular. Even in this case, there is some freedom in how to choose V_n; this is the only freedom that the Step B process provides when determining f_{n+1}. In any case, the crucial fact about V_n is that its blocks match those corresponding to distinct multiples of the identity that appear along the diagonal of D_n, implying $D_nV_n = V_nD_n$.

Having chosen V_n, we proceed to Step B.2. Here, we produce subsets $\mathcal{I}_n$ and $\mathcal{J}_n$ of $\{1, \dots, M\}$ that are the remnants of the indices of $\{\lambda_{n;m}\}_{m=1}^{M}$ and $\{\lambda_{n+1;m}\}_{m=1}^{M}$, respectively, obtained by canceling the values that are common to both sequences, working backwards from index M to index 1. An explicit algorithm for doing so is given in Table 3.2. Note that for each $m = M, \dots, 1$ (Line 03), we either remove a single element from both $\mathcal{I}_n^{(m)}$ and $\mathcal{J}_n^{(m)}$ (Lines 04–06) or remove nothing from both (Lines 07–09), meaning that $\mathcal{I}_n := \mathcal{I}_n^{(1)}$ and $\mathcal{J}_n := \mathcal{J}_n^{(1)}$ have the same cardinality, which we denote R_n. Moreover, since $\{\lambda_{n+1;m}\}_{m=1}^{M}$ interlaces on $\{\lambda_{n;m}\}_{m=1}^{M}$, then for any real scalar λ whose multiplicity as a value of $\{\lambda_{n;m}\}_{m=1}^{M}$ is L, we have that its multiplicity as a value of $\{\lambda_{n+1;m}\}_{m=1}^{M}$ is either $L-1$, L or $L+1$. When these two multiplicities are equal, this algorithm completely removes the corresponding indices from both $\mathcal{I}_n$ and $\mathcal{J}_n$. On the other hand, if the new multiplicity is $L-1$ or $L+1$,

then the least such index in $\mathcal{I}_n$ or $\mathcal{J}_n$ is left behind, respectively, leading to the definitions of $\mathcal{I}_n$ or $\mathcal{J}_n$ given in Step B.2. Having these sets, it is trivial to find the corresponding permutations $\pi_{\mathcal{I}_n}$ and $\pi_{\mathcal{J}_n}$ on $\{1,\ldots,M\}$ and to construct the associated projection matrices $\Pi_{\mathcal{I}_n}$ and $\Pi_{\mathcal{J}_n}$.

We now proceed to Step B.3. For the sake of notational simplicity, let $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ denote the values of $\{\lambda_{n;m}\}_{m\in\mathcal{I}_n}$ and $\{\lambda_{n+1;m}\}_{m\in\mathcal{J}_n}$, respectively. That is, let $\beta_{\pi_{\mathcal{I}_n}(m)} = \lambda_{n;m}$ for all $m \in \mathcal{I}_n$ and $\gamma_{\pi_{\mathcal{J}_n}(m)} = \lambda_{n+1;m}$ for all $m \in \mathcal{J}_n$. Note that due to the way in which $\mathcal{I}_n$ and $\mathcal{J}_n$ were defined, we have that the values of $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ are all distinct, both within each sequence and across the two sequences. Moreover, since $\{\lambda_{n;m}\}_{m\in\mathcal{I}_n}$ and $\{\lambda_{n+1;m}\}_{m\in\mathcal{J}_n}$ are nonincreasing while $\pi_{\mathcal{I}_n}$ and $\pi_{\mathcal{J}_n}$ are increasing on $\mathcal{I}_n$ and $\mathcal{J}_n$ respectively, then the values $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ are strictly decreasing.

We further claim that $\{\gamma_r\}_{r=1}^{R_n}$ interlaces on $\{\beta_r\}_{r=1}^{R_n}$. To see this, consider the four polynomials:

$$p_n(x) = \prod_{m=1}^{M}(x-\lambda_{n;m}), \qquad\qquad p_{n+1}(x) = \prod_{m=1}^{M}(x-\lambda_{n+1;m}),$$
$$b(x) = \prod_{r=1}^{R_n}(x-\beta_r), \qquad\qquad c(x) = \prod_{r=1}^{R_n}(x-\gamma_r). \tag{3.36}$$

Since $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ were obtained by canceling common terms from $\{\lambda_{n;m}\}_{m=1}^{M}$ and $\{\lambda_{n+1;m}\}_{m=1}^{M}$, we have that $p_{n+1}(x)/p_n(x) = c(x)/b(x)$ for all $x \notin \{\lambda_{n;m}\}_{m=1}^{M}$. Writing any $r = 1,\ldots,R_n$ as $r = \pi_{\mathcal{I}_n}(m)$ for some $m \in \mathcal{I}_n$, we have that since $\{\lambda_{n;m}\}_{m=1}^{M} \sqsubseteq \{\lambda_{n+1;m}\}_{m=1}^{M}$, applying the "only if" direction of Lemma 5 with "$p(x)$" and "$q(x)$" being $p_n(x)$ and $p_{n+1}(x)$ gives

$$\lim_{x\to\beta_r}(x-\beta_r)\frac{c(x)}{b(x)} = \lim_{x\to\lambda_{n;m}}(x-\lambda_{n;m})\frac{p_{n+1}(x)}{p_n(x)} \le 0. \tag{3.37}$$

Since (3.37) holds for all $r = 1,\ldots,R_n$, applying "if" direction of Lemma 5 with "$p(x)$" and "$q(x)$" being $b(x)$ and $c(x)$ gives that $\{\gamma_r\}_{r=1}^{R_n}$ indeed interlaces on $\{\beta_r\}_{r=1}^{R_n}$.

Taken together, the facts that $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ are distinct, strictly decreasing, and interlacing sequences implies that the $R_n \times 1$ vectors v_n and w_n are well-defined. To be

Table 3.2: An explicit algorithm for computing the index sets $\mathcal{I}_n$ and $\mathcal{J}_n$ in Step B.2 of Theorem 7.

$\mathcal{I}_n^{(M)} := \{1, \dots, M\}$
$\mathcal{J}_n^{(M)} := \{1, \dots, M\}$
for $m = M, \dots, 1$
  if $\lambda_{n;m} \in \{\lambda_{n+1;m'}\}_{m' \in \mathcal{J}_n^{(m)}}$
    $\mathcal{I}_n^{(m-1)} := \mathcal{I}_n^{(m)} \setminus \{m\}$
    $\mathcal{J}_n^{(m-1)} := \mathcal{J}_n^{(m)} \setminus \{m'\}$ where $m' = \max\{m'' \in \mathcal{J}_n^{(m)} : \lambda_{n+1;m''} = \lambda_{n;m}\}$
  else
    $\mathcal{I}_n^{(m-1)} := \mathcal{I}_n^{(m)}$
    $\mathcal{J}_n^{(m-1)} := \mathcal{J}_n^{(m)}$
  end if
end for
$\mathcal{I}_n := \mathcal{I}_n^{(1)}$
$\mathcal{J}_n := \mathcal{I}_n^{(1)}$

precise, Step B.3 may be rewritten as finding $v_n(r), w_n(r') \geq 0$ for all $r, r' = 1 \dots, R_n$ such that

$$[v_n(r)]^2 = -\frac{\prod_{r''=1}^{R_n}(\beta_r - \gamma_{r''})}{\prod_{\substack{r''=1 \\ r''\neq r}}^{R}(\beta_r - \beta_{r''})}, \qquad [w_n(r')]^2 = \frac{\prod_{r''=1}^{R_n}(\gamma_{r'} - \beta_{r''})}{\prod_{\substack{r''=1 \\ r''\neq r'}}^{R}(\gamma_{r'} - \gamma_{r''})}. \tag{3.38}$$

Note the fact that the β_r's and γ_r's are distinct implies that the denominators in (3.38) are nonzero, and moreover that the quotients themselves are nonzero. In fact, since $\{\beta_r\}_{r=1}^{R_n}$ is strictly decreasing, then for any fixed r, the values $\{\beta_r - \beta_{r''}\}_{r''\neq r}$ can be decomposed into $r-1$ negative values $\{\beta_r - \beta_{r''}\}_{r''=1}^{r-1}$ and $R_n - r$ positive values $\{\beta_r - \beta_{r''}\}_{r''=r+1}^{R_n}$. Moreover, since $\{\beta_r\}_{r=1}^{R_n} \sqsubseteq \{\gamma_r\}_{r=1}^{R_n}$, then for any such r, the values $\{\beta_r - \gamma_{r''}\}_{r''=1}^{R_n}$ can be broken into r

negative values $\{\beta_r - \gamma_{r''}\}_{r''=1}^{r}$ and $R_n - r$ positive values $\{\beta_r - \gamma_{r''}\}_{r''=r+1}^{R_n}$. With the inclusion of an additional negative sign, we see that the quantity defining $[v_n(r)]^2$ in (3.38) is indeed positive. Meanwhile, the quantity defining $[w_n(r')]^2$ has exactly $r' - 1$ negative values in both the numerator and denominator, namely $\{\gamma_{r'} - \beta_{r''}\}_{r''=1}^{r'-1}$ and $\{\gamma_{r'} - \gamma_{r''}\}_{r''=1}^{r'-1}$, respectively.

Having shown that the v_n and w_n of Step B.3 are well-defined, we now take f_{n+1} and U_{n+1} as defined in Steps B.4 and B.5. Recall that what remains to be shown in this direction of the proof is that U_{n+1} is a unitary matrix and that $F_{n+1} = \{f_{n'}\}_{n'=1}^{n+1}$ satisfies $F_{n+1}F_{n+1}^* U_{n+1} = U_{n+1}D_{n+1}$. To do so, consider the definition of U_{n+1} and recall that U_n is unitary by the inductive hypothesis, V_n is unitary by construction, and that the permutation matrices $\Pi_{\mathcal{I}_n}$ and $\Pi_{\mathcal{J}_n}$ are orthogonal, that is, unitary and real. As such, to show that U_{n+1} is unitary, it suffices to show that the $R_n \times R_n$ real matrix W_n is orthogonal. To do this, recall that eigenvectors corresponding to distinct eigenvalues of self-adjoint operators are necessarily orthogonal. As such, to show that W_n is orthogonal, it suffices to show that the columns of W_n are eigenvectors of a real symmetric operator. To this end, we claim

$$(D_{n;\mathcal{I}_n} + v_n v_n^{\mathrm{T}})W_n = W_n D_{n+1;\mathcal{J}_n}, \qquad W_n^{\mathrm{T}} W_n(r,r) = 1,\ \forall r = 1,\dots,R_n, \tag{3.39}$$

where $D_{n;\mathcal{I}_n}$ and $D_{n+1;\mathcal{J}_n}$ are the $R_n \times R_n$ diagonal matrices whose rth diagonal entries are given by $\beta_r = \lambda_{n;\pi_{\mathcal{I}_n}^{-1}(r)}$ and $\gamma_r = \lambda_{n+1;\pi_{\mathcal{J}_n}^{-1}(r)}$, respectively. To prove (3.39), note that for any $r, r' = 1,\dots,R_n$,

$$\begin{aligned}[(D_{n;\mathcal{I}_n} + v_n v_n^{\mathrm{T}})W_n](r,r') &= (D_{n;\mathcal{I}_n} W_n)(r,r') + (v_n v_n^{\mathrm{T}} W_n)(r,r') \\ &= \beta_r W_n(r,r') + v_n(r)\sum_{r''=1}^{R_n} v_n(r'')W_n(r'',r'). \end{aligned} \tag{3.40}$$

Rewriting the definition of W_n from Step B.5 in terms of $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ gives

$$W_n(r,r') = \frac{v_n(r)w_n(r')}{\gamma_{r'} - \beta_r}. \tag{3.41}$$

Substituting (3.41) into (3.40) gives

$$\begin{aligned}[(D_{n;\mathcal{I}_n} + v_n v_n^{\mathrm{T}})W_n](r, r') &= \beta_r \frac{v_n(r)w_n(r')}{\gamma_{r'} - \beta_r} + v_n(r) \sum_{r''=1}^{R_n} v_n(r'') \frac{v_n(r'')w_n(r')}{\gamma_{r'} - \beta_{r''}} \\ &= v_n(r)w_n(r') \left(\frac{\beta_r}{\gamma_{r'} - \beta_r} + \sum_{r''=1}^{R_n} \frac{[v_n(r'')]^2}{\gamma_{r'} - \beta_{r''}} \right). \end{aligned} \tag{3.42}$$

Simplifying (3.42) requires a polynomial identity. Note that the difference $\prod_{r''=1}^{R_n}(x - \gamma_{r''}) - \prod_{r''=1}^{R_n}(x - \beta_{r''})$ of two monic polynomials is itself a polynomial of degree at most $R_n - 1$, and as such it can be written as the Lagrange interpolating polynomial determined by the R_n distinct points $\{\beta_r\}_{r=1}^{R_n}$:

$$\begin{aligned}\prod_{r''=1}^{R_n}(x - \gamma_{r''}) - \prod_{r''=1}^{R_n}(x - \beta_{r''}) &= \sum_{r''=1}^{R_n} \left(\prod_{r=1}^{R_n}(\beta_{r''} - \gamma_r) - 0 \right) \prod_{\substack{r=1 \\ r \neq r''}}^{R_n} \frac{(x - \beta_r)}{(\beta_{r''} - \beta_r)} \\ &= \sum_{r''=1}^{R_n} \frac{\prod_{r=1}^{R_n}(\beta_{r''} - \gamma_r)}{\prod_{\substack{r=1 \\ r \neq r''}}^{R_n}(\beta_{r''} - \beta_r)} \prod_{\substack{r=1 \\ r \neq r''}}^{R_n}(x - \beta_r). \end{aligned} \tag{3.43}$$

Recalling the expression for $[v_n(r)]^2$ given in (3.38), (3.43) can be rewritten as

$$\prod_{r''=1}^{R_n}(x - \beta_{r''}) - \prod_{r''=1}^{R_n}(x - \gamma_{r''}) = \sum_{r''=1}^{R_n} [v_n(r'')]^2 \prod_{\substack{r=1 \\ r \neq r''}}^{R_n}(x - \beta_r). \tag{3.44}$$

Dividing both sides of (3.44) by $\prod_{r''=1}^{R_n}(x - \beta_{r''})$ gives

$$1 - \prod_{r''=1}^{R_n} \frac{(x - \gamma_{r''})}{(x - \beta_{r''})} = \sum_{r''=1}^{R_n} \frac{[v_n(r'')]^2}{(x - \beta_{r''})} \qquad \forall x \notin \{\beta_r\}_{r=1}^{R_n}. \tag{3.45}$$

For any $r' = 1, \ldots, R_n$, letting $x = \gamma_{r'}$ in (3.45) makes the left-hand product vanish, yielding the identity:

$$1 = \sum_{r''=1}^{R_n} \frac{[v_n(r'')]^2}{(\gamma_{r'} - \beta_{r''})} \qquad \forall r' = 1, \ldots, R_n. \tag{3.46}$$

Substituting (3.46) into (3.42) and then recalling (3.41) gives

$$\begin{aligned}
[(D_{n;\mathcal{I}_n} + v_n v_n^{\mathrm{T}})W_n](r,r') &= v_n(r)w_n(r')\Big(\frac{\beta_r}{\gamma_{r'} - \beta_r} + 1\Big) \\
&= \gamma_{r'}\frac{v_n(r)w_n(r')}{\gamma_{r'} - \beta_r} \\
&= \gamma_{r'} W_n(r,r') \\
&= (W_n D_{n+1;\mathcal{J}_n})(r,r').
\end{aligned} \tag{3.47}$$

As (3.47) holds for all $r, r' = 1, \dots, R_n$ we have the first half of our claim (3.39). In particular, we know that the columns of W_n are eigenvectors of the real symmetric operator $D_{n;\mathcal{I}_n} + v_n v_n^{\mathrm{T}}$ which correspond to the distinct eigenvalues $\{\gamma_r\}_{r=1}^{R_n}$. As such, the columns of W_n are orthogonal. To show that W_n is an orthogonal matrix, we must further show that the columns of W_n have unit norm, namely the second half of (3.39). To prove this, at any $x \notin \{\beta_r\}_{r=1}^{R_n}$ we differentiate both sides of (3.45) with respect to x to obtain

$$\sum_{r''=1}^{R_n}\Big[\prod_{\substack{r=1\\ r\neq r''}}^{R_n}\frac{(x-\gamma_r)}{(x-\beta_r)}\Big]\frac{\gamma_{r''} - \beta_{r''}}{(x-\beta_{r''})^2} = \sum_{r''=1}^{R_n}\frac{[v_n(r'')]^2}{(x-\beta_{r''})^2} \qquad \forall x \notin \{\beta_r\}_{r=1}^{R_n}. \tag{3.48}$$

For any $r' = 1, \dots, R_n$, letting $x = \gamma_{r'}$ in (3.48) makes the left-hand summands where $r'' \neq r'$ vanish; by (3.38), the remaining summand where $r'' = r'$ can be written as:

$$\frac{1}{[w_n(r')]^2} = \frac{\prod\limits_{\substack{r=1\\ r\neq r'}}^{R}(\gamma_{r'} - \gamma_r)}{\prod\limits_{r=1}^{R_n}(\gamma_{r'} - \beta_r)} = \Big[\prod_{\substack{r=1\\ r\neq r'}}^{R_n}\frac{(\gamma_{r'} - \gamma_r)}{(\gamma_{r'} - \beta_r)}\Big]\frac{\gamma_{r'} - \beta_{r'}}{(\gamma_{r'} - \beta_{r'})^2} = \sum_{r''=1}^{R_n}\frac{[v_n(r'')]^2}{(\gamma_{r'} - \beta_{r''})^2}. \tag{3.49}$$

We now use this identity to show that the columns of W_n have unit norm; for any $r' = 1, \ldots, R_n$, (3.41) and (3.49) give

$$\begin{aligned}
(W_n^{\mathrm{T}} W_n)(r', r') &= \sum_{r''=1}^{R_n} [W_n(r'', r')]^2 \\
&= \sum_{r''=1}^{R_n} \left(\frac{v_n(r'')w_n(r')}{\gamma_{r'} - \beta_{r''}} \right)^2 \\
&= [w_n(r')]^2 \sum_{r''=1}^{R_n} \frac{[v_n(r'')]^2}{(\gamma_{r'} - \beta_{r''})^2} \\
&= [w_n(r')]^2 \frac{1}{[w_n(r')]^2} \\
&= 1.
\end{aligned}$$

Having shown that W_n is orthogonal, we have that U_{n+1} is unitary. For this direction of the proof, all that remains to be shown is that $F_{n+1}F_{n+1}^* U_{n+1} = U_{n+1} D_{n+1}$. To do this, we write $F_{n+1}F_{n+1}^* = F_n F_n^* + f_{n+1} f_{n+1}^*$ and recall the definition of U_{n+1}:

$$\begin{aligned}
F_{n+1}F_{n+1}^* U_{n+1} &= (F_n F_n^* + f_{n+1} f_{n+1}^*) U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} \\
&= F_n F_n^* U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} + f_{n+1} f_{n+1}^* U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n}. \qquad (3.50)
\end{aligned}$$

To simplify the first term in (3.50), recall that the inductive hypothesis gives $F_n F_n^* U_n = U_n D_n$ and that V_n was constructed to satisfy $D_n V_n = V_n D_n$, implying

$$\begin{aligned}
F_n F_n^* U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} &= U_n V_n D_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} \\
&= U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} (\Pi_{\mathcal{I}_n} D_n \Pi_{\mathcal{I}_n}^{\mathrm{T}}) \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n}. \qquad (3.51)
\end{aligned}$$

To continue simplifying (3.51), note that $\Pi_{\mathcal{I}_n} D_n \Pi_{\mathcal{I}_n}^{\mathrm{T}}$ is itself a diagonal matrix: for any $m, m' = 1, \dots, M$, the definition of the permutation matrix $\Pi_{\mathcal{I}_n}$ given in Step B.2 gives

$$\begin{aligned}(\Pi_{\mathcal{I}_n} D_n \Pi_{\mathcal{I}_n}^{\mathrm{T}})(m, m') &= \langle D_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \delta_{m'}, \Pi_{\mathcal{I}_n}^{\mathrm{T}} \delta_m \rangle \\ &= \langle D_n \delta_{\pi_{\mathcal{I}_n}^{-1}(m')}, \delta_{\pi_{\mathcal{I}_n}^{-1}(m)} \rangle = \begin{cases} \lambda_{n;\pi_{\mathcal{I}_n}^{-1}(m)}, & m = m', \\ 0, & m \neq m'. \end{cases}\end{aligned}$$

That is, $\Pi_{\mathcal{I}_n} D_n \Pi_{\mathcal{I}_n}^{\mathrm{T}}$ is the diagonal matrix whose first R_n diagonal entries $\{\beta_r\}_{r=1}^{R_n} = \{\lambda_{n;\pi_{\mathcal{I}_n}^{-1}(r)}\}_{r=1}^{R_n}$ match those of the aforementioned $R_n \times R_n$ diagonal matrix $D_{n;\mathcal{I}_n}$ and whose remaining $M - R_n$ diagonal entries $\{\lambda_{n;\pi_{\mathcal{I}_n}^{-1}(m)}\}_{m=R_n+1}^{M}$ form the diagonal of an $(M - R_n) \times (M - R_n)$ diagonal matrix $D_{n;\mathcal{I}_n^{\mathrm{c}}}$:

$$\Pi_{\mathcal{I}_n} D_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} = \begin{bmatrix} D_{n;\mathcal{I}_n} & 0 \\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}} \end{bmatrix}. \tag{3.52}$$

Substituting (3.52) into (3.51) gives

$$\begin{aligned}F_n F_n^* U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} &= U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} D_{n;\mathcal{I}_n} & 0 \\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}} \end{bmatrix} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} \\ &= U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} D_{n;\mathcal{I}_n} W_n & 0 \\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}} \end{bmatrix} \Pi_{\mathcal{J}_n}. \end{aligned} \tag{3.53}$$

To simplify the second term in (3.50), we recall the definition of f_{n+1} from Step B.4:

$$\begin{aligned}f_{n+1} f_{n+1}^* U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} &= U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} v_n \\ 0 \end{bmatrix} \begin{bmatrix} v_n^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} W_n & 0 \\ 0 & \mathrm{I} \end{bmatrix} \Pi_{\mathcal{J}_n} \\ &= U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} v_n v_n^{\mathrm{T}} W_n & 0 \\ 0 & 0 \end{bmatrix} \Pi_{\mathcal{J}_n}. \end{aligned} \tag{3.54}$$

Substituting (3.53) and (3.54) into (3.50), simplifying the result, and recalling (3.39) gives

$$\begin{aligned} F_{n+1}F_{n+1}^*U_{n+1} &= U_nV_n\Pi_{\mathcal{I}_n}^{\mathrm{T}}\begin{bmatrix}(D_{n;\mathcal{I}_n}+v_nv_n^{\mathrm{T}})W_n & 0\\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}}\end{bmatrix}\Pi_{\mathcal{J}_n}\\ &= U_nV_n\Pi_{\mathcal{I}_n}^{\mathrm{T}}\begin{bmatrix}W_nD_{n+1;\mathcal{J}_n} & 0\\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}}\end{bmatrix}\Pi_{\mathcal{J}_n}. \end{aligned}$$

By introducing an extra permutation matrix and its inverse and recalling the definition of U_{n+1}, this simplifies to

$$\begin{aligned} F_{n+1}F_{n+1}^*U_{n+1} &= U_nV_n\Pi_{\mathcal{I}_n}^{\mathrm{T}}\begin{bmatrix}W_n & 0\\ 0 & \mathrm{I}\end{bmatrix}\Pi_{\mathcal{J}_n}\Pi_{\mathcal{J}_n}^{\mathrm{T}}\begin{bmatrix}D_{n+1;\mathcal{J}_n} & 0\\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}}\end{bmatrix}\Pi_{\mathcal{J}_n}\\ &= U_{n+1}\Pi_{\mathcal{J}_n}^{\mathrm{T}}\begin{bmatrix}D_{n+1;\mathcal{J}_n} & 0\\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}}\end{bmatrix}\Pi_{\mathcal{J}_n}. \end{aligned} \tag{3.55}$$

We now partition the $\{\lambda_{n+1;m}\}_{m=1}^M$ of D_{n+1} into $\mathcal{J}_n$ and $\mathcal{J}_n^{\mathrm{c}}$ and mimic the derivation of (3.52), writing D_{n+1} in terms of $D_{n+1;\mathcal{J}_n}$ and $D_{n+1;\mathcal{J}_n^{\mathrm{c}}}$. Note here that by the manner in which $\mathcal{I}_n$ and $\mathcal{J}_n$ were constructed, the values of $\{\lambda_{n;m}\}_{m\in\mathcal{I}_n^{\mathrm{c}}}$ are equal to those of $\{\lambda_{n+1;m}\}_{\mathcal{J}_n^{\mathrm{c}}}$, as the two sets represent exactly those values which are common to both $\{\lambda_{n;m}\}_{m=1}^M$ and $\{\lambda_{n+1;m}\}_{m=1}^M$. As these two sequences are also both in nonincreasing order, we have $D_{n;\mathcal{I}_n^{\mathrm{c}}} = D_{n+1;\mathcal{J}_n^{\mathrm{c}}}$ and so

$$\Pi_{\mathcal{J}_n}D_{n+1}\Pi_{\mathcal{J}_n}^{\mathrm{T}} = \begin{bmatrix}D_{n+1;\mathcal{J}_n} & 0\\ 0 & D_{n+1;\mathcal{J}_n^{\mathrm{c}}}\end{bmatrix} = \begin{bmatrix}D_{n+1;\mathcal{J}_n} & 0\\ 0 & D_{n;\mathcal{I}_n^{\mathrm{c}}}\end{bmatrix}. \tag{3.56}$$

Substituting (3.56) into (3.55) yields $F_{n+1}F_{n+1}^*U_{n+1} = U_{n+1}D_{n+1}$, completing this direction of the proof.

($\Rightarrow$) Let $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ be any nonnegative nonincreasing sequences, and let $F = \{f_n\}_{n=1}^N$ be any sequence of vectors whose frame operator FF^* has $\{\lambda_m\}_{m=1}^M$ as its spectrum and has $\|f_n\|^2 = \mu_n$ for all $n = 1,\dots,N$. We will show that this F can be constructed by following Step A and Step B of this result. To see this, for any $n = 1,\dots,N$, let $F_n = \{f_{n'}\}_{n'=1}^n$ and let $\{\lambda_{n;m}\}_{m=1}^M$ be the spectrum of the corresponding frame operator

$F_n F_n^*$. Letting $\lambda_{0;m} := 0$ for all m, the proof of Theorem 2 demonstrated that the sequence of spectra $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ necessarily forms a sequence of outer eigensteps as specified by Definition 1. This particular set of outer eigensteps is the one we choose in Step A.

All that remains to be shown is that we can produce our specific F by using Step B. Here, we must carefully exploit our freedom to pick U_1 and the V_n's; the proper choice of these unitary matrices will result in F, while other choices will produce other sequences of vectors that are only related to F through a potentially complicated series of rotations. Indeed, note that since $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ is a valid sequence of outer eigensteps, then the other direction of this proof, as given earlier, implies that any choice of U_1 and V_n's will result in a sequence of vectors whose outer eigensteps match those of F. Moreover, quantities that we considered in the other direction of the proof that only depended on the choice of outer eigensteps, such as $\mathcal{I}_n$, $\mathcal{J}_n$, $\{\beta_r\}_{r=1}^{R_n}$, $\{\gamma_r\}_{r=1}^{R_n}$, etc., are thus also well-defined in this direction; in the following arguments, we recall several such quantities and make further use of their previously-derived properties.

To be precise, let U_1 be any one of the infinite number of unitary matrices whose first column $u_{1;1}$ satisfies $f_1 = \sqrt{\mu_1}u_{1;1}$. We now proceed by induction, assuming that for any given $n = 1, \dots, N-1$, we have followed Step B and have made appropriate choices for $\{V_{n'}\}_{n'=1}^{n-1}$ so as to correctly produce $F_n = \{f_{n'}\}_{n'=1}^{n}$; we show how the appropriate choice of V_n will correctly produce f_{n+1}. To do so, we again write the nth spectrum $\{\lambda_{n;m}\}_{m=1}^{M}$ in terms of its multiplicities as $\{\lambda_{n;m(k,l)}\}_{k=1}^{K_n}{}_{l=1}^{L_{n;k}}$. For any $k = 1, \dots, K_n$, Step B of Theorem 2 gives that the norm of the projection of f_{n+1} onto the kth eigenspace of $F_n F_n^*$ is necessarily given by

$$\|P_{n;\lambda_{n;m(k,1)}} f_{n+1}\|^2 = -\lim_{x \to \lambda_{n;m(k,1)}} (x - \lambda_{n;m(k,1)}) \frac{p_{n+1}(x)}{p_n(x)}, \tag{3.57}$$

where $p_n(x)$ and $p_{n+1}(x)$ are defined by (3.36). Note that by picking $l = 1$, $\lambda_{n;m(k,1)}$ represents the first appearance of that particular value in $\{\lambda_{n;m}\}_{m=1}^{M}$. As such, these indices are the only ones that are eligible to be members of the set $\mathcal{I}_n$ found in Step B.2. That

is, $\mathcal{I}_n \subseteq \{m(k,1) : k = 1, \dots, K_n\}$. However, these two sets of indices are not necessarily equal, since $\mathcal{I}_m$ only contains m's of the form $m(k,1)$ that satisfy the additional property that the multiplicity of $\lambda_{n;m}$ as a value in $\{\lambda_{n;m'}\}_{m'=1}^{M}$ exceeds its multiplicity as a value in $\{\lambda_{n+1;m}\}_{m=1}^{M}$. To be precise, for any given $k = 1, \dots, K_n$, if $m(k,1) \in \mathcal{I}_n^{\mathrm{c}}$ then $\lambda_{n;m(k,1)}$ appears as a root of $p_{n+1}(x)$ at least as many times as it appears as a root of $p_n(x)$, meaning in this case that the limit in (3.57) is necessarily zero. If, on the other hand, $m(k,1) \in \mathcal{I}_n$, then writing $\pi_{\mathcal{I}_n}(m(k,1))$ as some $r \in \{1, \dots, R_n\}$ and recalling the definitions of $b(x)$ and $c(x)$ in (3.36) and $v(r)$ in (3.38), we can rewrite (3.57) as

$$\|P_{n;\beta_r} f_{n+1}\|^2 = -\lim_{x\to\beta_r}(x-\beta_r)\frac{p_{n+1}(x)}{p_n(x)} = -\lim_{x\to\beta_r}(x-\beta_r)\frac{c(x)}{b(x)} = -\frac{\displaystyle\prod_{r''=1}^{R_n}(\beta_r-\gamma_{r''})}{\displaystyle\prod_{\substack{r''=1\\ r''\neq r}}^{R}(\beta_r-\beta_{r''})} = [v_n(r)]^2. \quad (3.58)$$

As such, we can write f_{n+1} as

$$\begin{aligned} f_{n+1} &= \sum_{k=1}^{K_n} P_{n;\lambda_{n;m(k,1)}} f_{n+1} \\ &= \sum_{r=1}^{R_n} P_{n;\beta_r} f_{n+1} \\ &= \sum_{r=1}^{R_n} v_n(r)\frac{1}{v_n(r)} P_{n;\beta_r} f_{n+1} \\ &= \sum_{m\in\mathcal{I}_n} v_n(\pi_{\mathcal{I}_n}(m))\frac{1}{v_n(\pi_{\mathcal{I}_n}(m))} P_{n;\beta_{\pi_{\mathcal{I}_n}(m)}} f_{n+1} \end{aligned} \quad (3.59)$$

where each $\frac{1}{v_n(\pi_{\mathcal{I}_n}(m))} P_{n;\beta_{\pi_{\mathcal{I}_n}(m)}} f_{n+1}$ has unit norm by (3.58). We now pick a new orthonormal eigenbasis $\hat{U}_n := \{\hat{u}_{n;m}\}_{m=1}^{M}$ for $F_n F_n^*$ that has the property that for any $k = 1, \dots, K_n$, both $\{u_{n;m(k,l)}\}_{l=1}^{L_{n;k}}$ and $\{\hat{u}_{n;m(k,l)}\}_{l=1}^{L_{n;k}}$ span the same eigenspace and, for every $m(k,1) \in \mathcal{I}_n$, has the

additional property that $\hat{u}_{n;m(k,1)} = \frac{1}{v_n(\pi_{\mathcal{I}_n}(m(k,1)))} P_{n;\beta_{\pi_{\mathcal{I}_n}(m(k,1))}} f_{n+1}$. As such, (3.59) becomes

$$\begin{aligned} f_{n+1} &= \sum_{m \in \mathcal{I}_n} v_n(\pi_{\mathcal{I}_n}(m)) \hat{u}_{n;m} \\ &= \hat{U}_n \sum_{m \in \mathcal{I}_n} v_n(\pi_{\mathcal{I}_n}(m)) \delta_m \\ &= \hat{U}_n \sum_{r=1}^{R_n} v_n(r) \delta_{\pi_{\mathcal{I}_n}^{-1}(r)} \\ &= \hat{U}_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \sum_{r=1}^{R_n} v_n(r) \delta_r \\ &= \hat{U}_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} v_n \\ 0 \end{bmatrix}. \end{aligned} \tag{3.60}$$

Letting V_n be the unitary matrix $V_n = U_n^* \hat{U}_n$, the eigenspace spanning condition gives that V_n is block-diagonal whose kth diagonal block is of size $L_{n;k} \times L_{n;k}$. Moreover, with this choice of V_n, (3.60) becomes

$$f_{n+1} = U_n U_n^* \hat{U}_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} v_n \\ 0 \end{bmatrix} = U_n V_n \Pi_{\mathcal{I}_n}^{\mathrm{T}} \begin{bmatrix} v_n \\ 0 \end{bmatrix}$$

meaning that f_{n+1} can indeed be constructed by following Step B. □

IV. Top Kill algorithm and the characterization of the set of all eigensteps

In this chapter, we continue the work of Chapter 3 and show how to explicitly construct every sequence of eigensteps—a sequence of interlacing spectra—as required by Step A of Theorem 2. The major results are Theorem 16 which provides the explicit Top Kill algorithm for constructing a sequence of eigensteps, and Theorem 17 which gives a complete characterization of the set of all eigensteps for a given $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$. When these results are combined with those in Chapter 3, they provide a complete solution to the following problem:

Problem 9. *Given any nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$, construct all $F = \{f_n\}_{n=1}^N$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^M$ and for which $\|f_n\|^2 = \mu_n$ for all n.*

To proceed, we first recall that any $F = \{f_n\}_{n=1}^N$ for which FF^* has $\{\lambda_m\}_{m=1}^M$ as its spectrum and for which $\|f_n\|^2 = \mu_n$ for all n generates a sequence of outer eigensteps. In Chapter 3, Theorem 2 proves that the converse of this statement is also true. As already noted, Theorem 2 is not an easily-implementable algorithm. While we have already seen that Step B can be made more explicit (Theorem 7), Step A is still rather vague. The techniques in this chapter will make Step A explicit, with our main result being:

Theorem 10. *Let $\{\lambda_m\}_{m=1}^M$ and $\{\mu_n\}_{n=1}^N$ be nonnegative and nonincreasing where $M \leq N$. There exists a sequence of vectors $F = \{f_n\}_{n=1}^N$ in $\mathbb{C}^M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^M$ and for which $\|f_n\|^2 = \mu_n$ for all n if and only if $\{\lambda_m\}_{m=1}^M \cup \{0\}_{m=M+1}^N \succeq \{\mu_n\}_{n=1}^N$. Moreover, if $\{\lambda_m\}_{m=1}^M \cup \{0\}_{m=M+1}^N \succeq \{\mu_n\}_{n=1}^N$, then every such F can be constructed by the following process:*

A. *Let $\{\lambda_{N;m}\}_{m=1}^M := \{\lambda_m\}_{m=1}^M$.*

   For $n = N, \dots, 2$, construct $\{\lambda_{n-1;m}\}_{m=1}^M$ in terms of $\{\lambda_{n;m}\}_{m=1}^M$ as follows:

For each $k = M, \dots, 1$, if $k > n-1$, take $\lambda_{n-1;k} := 0$.

Otherwise, pick any $\lambda_{n-1;k} \in [A_{n-1;k}, B_{n-1;k}]$, where

$$A_{n-1;k} := \max\left\{\lambda_{n;k+1}, \sum_{m=k}^{M} \lambda_{n;m} - \sum_{m=k+1}^{M} \lambda_{n-1;m} - \mu_n\right\},$$

$$B_{n-1;k} := \min\left\{\lambda_{n;k}, \min_{l=1,\dots,k}\left\{\sum_{m=l}^{n-1} \mu_m - \sum_{m=l+1}^{k} \lambda_{n;m} - \sum_{m=k+1}^{M} \lambda_{n-1;m}\right\}\right\}.$$

Here, we use the convention that $\lambda_{n;M+1} = 0$, and that sums over empty sets of indices are zero.

B. *Follow Step B of Theorem 2.*

Conversely, any F constructed by this process has $\{\lambda_m\}_{m=1}^M$ as the spectrum of FF^* and $\|f_n\|^2 = \mu_n$ for all n, and moreover, $F_nF_n^*$ has spectrum $\{\lambda_{n;m}\}_{m=1}^M$.

In Section 4.1, we begin by discussing how solving Problem 9 via Theorem 2 suffices to construct all Schur-Horn matrices. We also introduce the notion of *inner eigensteps*. While outer eigensteps give the spectra of the partial frame operators, inner eigensteps will give the spectra of the partial Gram matrices $F_n^*F_n$. In Section 4.2, we visualize the inner eigenstep construction problem in terms of iteratively building a staircase. We then provide a new algorithm, called *Top Kill*, which produces a sequence of inner eigensteps whenever possible to do so. Finally, in Section 4.3, we find an explicit parametrization of the set of all valid inner eigensteps for a given $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$.

4.1 Preliminaries

In this section, we first further detail the connection between the Schur-Horn Theorem and Problem 9. We then reformulate Step A of Theorem 2 in terms of *inner eigensteps* which are an alternative, but mathematically equivalent version of the outer eigensteps defined in Chapter 3; it turns out that it is more convenient to work with spectra whose number of elements equals that of $\{\mu_n\}_{n=1}^N$, namely N.

By applying the Schur-Horn Theorem (page 15) to the Gram matrix F^*F, whose diagonal entries are given by $\{\|f_n\|^2\}_{n=1}^N$ and whose spectrum is $\{\lambda_n\}_{n=1}^N$, we are able to determine when a frame with prescribed spectrum and lengths exists. Here it's important to note that the spectrum of the Gram matrix is a zero-padded version of the spectrum $\{\lambda_m\}_{m=1}^M$ of the frame operator FF^*. The only difference between the eigenvalues of the Gram matrix and the frame operator is zero eigenvalues; that is, $\lambda_{n;m} = 0$ for $M < n \leq N$. Thus the Schur-Horn Theorem implies that Problem 9 is feasible if and only if $\{\mu_n\}_{n=1}^N$ is majorized by $\{\lambda_m\}_{m=1}^M$ padded with $N - M$ zeros. Indeed, if Problem 9 has a solution F, then $G = F^*F$ has spectrum $\{\lambda_m\}_{m=1}^M \cup \{0\}_{m=M+1}^N$ and diagonal $\{\mu_n\}_{n=1}^N$, and so $\{\mu_n\}_{n=1}^N \preceq \{\lambda_m\}_{m=1}^M \cup \{0\}_{m=M+1}^N$ by the Schur-Horn Theorem. Conversely, if $\{\mu_n\}_{n=1}^N \preceq \{\lambda_m\}_{m=1}^M \cup \{0\}_{m=M+1}^N$ then the corresponding Schur-Horn matrix G can be unitarily diagonalized:

$$G = VDV^* = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} = V_1 D_1 V_1^*,$$

where D_1 is an $M \times M$ diagonal matrix with diagonal $\{\lambda_m\}_{m=1}^M$; the matrix $F = D_1^{\frac{1}{2}} V_1^*$ is then one solution to Problem 9. This line of reasoning is well-known [1, 19].

We follow an alternative approach that is modeled in that of [33]: rather than use the Schur-Horn Theorem to determine the feasibility of Problem 9, we instead independently find all solutions to Problem 9—see Theorem 10—and then use these matrices to construct all Schur-Horn matrices. To be precise, note that though the Schur-Horn Theorem applies to all self-adjoint matrices G, it suffices to consider the case where G is positive semidefinite. Indeed, any self-adjoint matrix $\hat{G}$ can be written as $\hat{G} = G + \alpha \mathrm{I}$ where G is positive semidefinite and $\alpha \leq \lambda_{\min}(\hat{G})$; it is straightforward to show that the spectrum $\{\hat{\lambda}_n\}_{n=1}^N$ of $\hat{G}$ majorizes its diagonal $\{\hat{\mu}_n\}_{n=1}^N$ if and only if the spectrum $\{\lambda_n\}_{n=1}^N = \{\hat{\lambda}_n - \alpha\}_{n=1}^N$ of G majorizes its diagonal $\{\mu_n\}_{n=1}^N = \{\hat{\mu}_n - \alpha\}_{n=1}^N$. Moreover, since G is positive semidefinite, it has a Cholesky factorization $G = F^*F$ where $F \in \mathbb{C}^{N \times N}$. Regarding F as the synthesis operator of some sequence of vectors $\{f_n\}_{n=1}^N$ in $\mathbb{C}^N$, we are thus reduced to Problem 9 in

the special case where $M = N$. Presuming for the moment that Theorem 10 is true, we summarize the above discussion as follows:

Theorem 11. *Given nonincreasing sequences $\{\hat{\lambda}_n\}_{n=1}^{N}$ and $\{\hat{\mu}_n\}_{n=1}^{N}$ such that $\{\hat{\lambda}_n\}_{n=1}^{N}$ majorizes $\{\hat{\mu}_n\}_{n=1}^{N}$, every self-adjoint matrix $\hat{G}$ with spectrum $\{\hat{\lambda}_n\}_{n=1}^{N}$ and diagonal $\{\hat{\mu}_n\}_{n=1}^{N}$ can be constructed as $\hat{G} = F^*F + \alpha\mathrm{I}$ where F is any matrix constructed by taking any $\alpha \leq \lambda_{\min}(\hat{G})$ and applying Theorem 10 where $\lambda_n := \hat{\lambda}_n - \alpha$ and $\mu_n := \hat{\mu}_n - \alpha$. Moreover, any $\hat{G}$ constructed in this fashion has the desired spectrum and diagonal.*

The remainder of this chapter is focused on solving Problem 9. In particular, we focus on Part A of Theorem 2 of finding a valid sequence of outer eigensteps for any given nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$. Recall Example 8 from Chapter 3. While solving for the conditions upon the eigensteps was fairly simple in that example, its complexity will quickly increase as M and N get larger. Finding all valid sequences of eigensteps (3.26) often requires reducing a large system of linear inequalities (3.27). The goal of the remaining sections of this chapter is to derive a more efficient algorithm for systematically finding the necessary conditions upon a given sequence of eigensteps. It turns out that this method is more easily understood in terms of an alternative but equivalent notion of eigensteps. To be clear, for any given sequence of outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$, recall from Theorem 2 that for any $n = 1, \ldots, N$, the sequence $\{\lambda_{n;m}\}_{m=1}^{M}$ is the spectrum of the $M \times M$ frame operator (3.4) of the nth partial sequence $F_n = \{f_{n'}\}_{n'=1}^{n}$. In the theory that follows, it is more convenient to instead work with the spectrum $\{\lambda_{n;m}\}_{m=1}^{n}$ of the corresponding $n \times n$ Gram matrix $F_n^*F_n$; we use the same notation for both spectra since $\{\lambda_{n;m}\}_{m=1}^{n}$ is a zero-padded version of $\{\lambda_{n;m}\}_{m=1}^{M}$ or vice versa, depending on whether $n > M$ or $n \leq M$. We refer to the values $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$ as a sequence of *inner eigensteps* since they arise from matrices of inner products of the f_n's (Gram matrices), whereas outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ arise from sums of outer products of the f_n's (frame operators). To be precise:

Definition 12. Let $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$ be nonnegative nonincreasing sequences. A corresponding sequence of *inner eigensteps* is a sequence of sequences $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=1}^N$ which satisfies the following three properties:

(i) The final sequence is $\{\lambda_m\}_{m=1}^M$: $\lambda_{N;m} = \lambda_m$ for every $m = 1, \dots, N$,

(ii) The sequences interlace: $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^n$ for every $n = 2, \dots, N$,

(iii) The trace condition is satisfied: $\sum_{m=1}^n \lambda_{n;m} = \sum_{m=1}^n \mu_m$ for every $n = 1, \dots, N$.

Unlike the outer eigensteps of Definition 1, the interlacing relation (ii) here involves two sequences of different length; we write $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^n$ if $\beta_{m+1} \leq \alpha_m \leq \beta_m$ for all $m = 1, \dots, n-1$. Now we revisit Example 8 to demonstrate the connection between inner and outer eigensteps:

Example 13. We revisit Example 8. Here, we pad $\{\lambda_m\}_{m=1}^3$ with two zeros so as to match the length of $\{\mu_n\}_{n=1}^5$. That is, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{5}{3}$, $\lambda_4 = \lambda_5 = 0$, and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 1$. We find every sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=1}^5$, namely every table of the form:

n	1	2	3	4	5
$\lambda_{n;5}$					0
$\lambda_{n;4}$				?	0
$\lambda_{n;3}$			?	?	$\frac{5}{3}$
$\lambda_{n;2}$		?	?	?	$\frac{5}{3}$
$\lambda_{n;1}$	?	?	?	?	$\frac{5}{3}$

(4.1)

that satisfies the interlacing properties (ii) and trace conditions (iii) of Definition 12. To be precise, (ii) gives us $0 = \lambda_{5;5} \leq \lambda_{4;4} \leq \lambda_{5;4} = 0$ and so $\lambda_{4;4} = 0$. Similarly,

$\frac{5}{3} \leq \lambda_{5;3} \leq \lambda_{4;2} \leq \lambda_{3;1} \leq \lambda_{4;1} \leq \lambda_{5;1} = \frac{5}{3}$ and so $\lambda_{4;2} = \lambda_{3;1} = \lambda_{4;1} = \frac{5}{3}$, yielding:

$$\begin{array}{cccccc}
n & 1 & 2 & 3 & 4 & 5 \\
\hline
\lambda_{n;5} & & & & & 0 \\
\lambda_{n;4} & & & & 0 & 0 \\
\lambda_{n;3} & & & ? & ? & \frac{5}{3} \\
\lambda_{n;2} & & ? & ? & \frac{5}{3} & \frac{5}{3} \\
\lambda_{n;1} & ? & ? & \frac{5}{3} & \frac{5}{3} & \frac{5}{3}
\end{array} \tag{4.2}$$

Meanwhile, since $\mu_m = 1$ for all m, the trace conditions (iii) give that the values in the nth column of (4.2) sum to n. Thus, $\lambda_{1;1} = 1$ and $\lambda_{4;3} = \frac{2}{3}$:

$$\begin{array}{cccccc}
n & 1 & 2 & 3 & 4 & 5 \\
\hline
\lambda_{n;5} & & & & & 0 \\
\lambda_{n;4} & & & & 0 & 0 \\
\lambda_{n;3} & & & ? & \frac{2}{3} & \frac{5}{3} \\
\lambda_{n;2} & & ? & ? & \frac{5}{3} & \frac{5}{3} \\
\lambda_{n;1} & 1 & ? & \frac{5}{3} & \frac{5}{3} & \frac{5}{3}
\end{array}$$

To proceed, we label $\lambda_{3;3}$ as x and $\lambda_{2;2}$ as y, at which point (iii) uniquely determines $\lambda_{3;2}$ and $\lambda_{2;1}$:

$$\begin{array}{cccccc}
n & 1 & 2 & 3 & 4 & 5 \\
\hline
\lambda_{n;5} & & & & & 0 \\
\lambda_{n;4} & & & & 0 & 0 \\
\lambda_{n;3} & & & x & \frac{2}{3} & \frac{5}{3} \\
\lambda_{n;2} & & y & \frac{4}{3} - x & \frac{5}{3} & \frac{5}{3} \\
\lambda_{n;1} & 1 & 2 - y & \frac{5}{3} & \frac{5}{3} & \frac{5}{3}
\end{array} \tag{4.3}$$

For our particular choice of $\{\lambda_n\}_{n=1}^5$ and $\{\mu_n\}_{n=1}^5$, the above argument shows that every corresponding sequence of inner eigensteps is of the form (4.3). Conversely, one may immediately verify that any $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=1}^5$ of this form satisfies (i) and (iii) of Definition 12 and moreover satisfies (ii) when $n = 5$. However, in order to satisfy (ii) for $n = 2, 3, 4$, x and y must be chosen so that they satisfy the ten inequalities:

$$\begin{aligned} \{\lambda_{3;m}\}_{m=1}^3 \sqsubseteq \{\lambda_{4;m}\}_{m=1}^4 \quad &\Longleftrightarrow \quad 0 \le x \le \tfrac{2}{3} \le \tfrac{4}{3} - x \le \tfrac{5}{3}, \\ \{\lambda_{2;m}\}_{m=1}^2 \sqsubseteq \{\lambda_{3;m}\}_{m=1}^3 \quad &\Longleftrightarrow \quad x \le y \le \tfrac{4}{3} - x \le 2 - y \le \tfrac{5}{3}, \\ \{\lambda_{1;m}\}_{m=1}^1 \sqsubseteq \{\lambda_{2;m}\}_{m=1}^2 \quad &\Longleftrightarrow \quad y \le 1 \le 2 - y. \end{aligned} \tag{4.4}$$

A quick inspection reveals the system (4.4) to be equivalent to the one derived in the outer eigenstep formulation (3.27) presented in Example 8, which is reducible to $0 \le x \le \frac{2}{3}$, $\max\{\frac{1}{3}, x\} \le y \le \min\{\frac{2}{3} + x, \frac{4}{3} - x\}$. Moreover, we see that the outer eigensteps (3.26) that arise from $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\}$ and the inner eigensteps (4.3) that arise from $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} = \{\frac{5}{3}, \frac{5}{3}, \frac{5}{3}, 0, 0\}$ are but zero-padded versions of each other; the next result claims that such a result holds in general.

Theorem 14. *Let $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$ be nonnegative and nonincreasing, and choose any $M \le N$ such that $\lambda_n = 0$ for every $n > M$. Then, every choice of outer eigensteps (Definition 1) corresponds to a unique choice of inner eigensteps (Definition 12) and vice versa, the two being zero-padded versions of each other.*

Specifically, a sequence of outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ gives rise to a sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=1}^N$, where $\lambda_{n;m} := 0$ whenever $m > M$. Conversely, a sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=1}^N$ gives rise to a sequence of outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$, where $\lambda_{n;m} := 0$ whenever $m > n$.

Moreover, $\{\lambda_{n;m}\}_{m=1}^M$ is the spectrum of the frame operator $F_n F_n^*$ of $F_n = \{f_m\}_{m=1}^n$ if and only if $\{\lambda_{n;m}\}_{m=1}^n$ is the spectrum of the Gram matrix $F_n^* F_n$.

Proof. First, take outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$, and consider $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$, where we define

$$\lambda_{n;m} = 0 \qquad \text{whenever } m > M. \tag{4.5}$$

Then Definition 12.(i) follows from Definition 1.(ii) in the case when $m \leq M$, and follows from (4.5) and the assumption that $\lambda_n = 0$ for every $n > M$ in the case when $m > M$. Next, we note that Definition 1.(iii) gives

$$\lambda_{n;m+1} \leq \lambda_{n-1;m} \leq \lambda_{n;m} \qquad \forall m = 1, \ldots, M-1, \tag{4.6}$$

$$0 \leq \lambda_{n-1;M} \leq \lambda_{n;M}, \tag{4.7}$$

for every $n = 1, \ldots, N$. To prove Definition 12.(ii), pick any $n = 2, \ldots, N$. We need to show $\lambda_{n;m+1} \leq \lambda_{n-1;m} \leq \lambda_{n;m}$ for every $m = 1, \ldots, n-1$. This follows directly from (4.6) when $n \leq M$ or when $n > M$ and $m < M$. If $n > M$ and $m = M$, then (4.5) and (4.7) together give

$$\lambda_{n;M+1} = 0 \leq \lambda_{n-1;M} \leq \lambda_{n;M}.$$

Also, (4.5) gives that $\lambda_{n;m+1} \leq \lambda_{n-1;m} \leq \lambda_{n;m}$ becomes $0 \leq 0 \leq 0$ whenever $n > M$ and $m > M$. Next, to show that $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$ satisfies Definition 12.(iii), note that when $n \geq M$, (4.5) and Definition 1.(iv) together give

$$\sum_{m=1}^{n} \lambda_{n;m} = \sum_{m=1}^{M} \lambda_{n;m} = \sum_{m=1}^{n} \mu_m. \tag{4.8}$$

Furthermore, if $n < M$, then Definition 1.(i) and Definition 1.(iii) together give

$$\lambda_{n;m} = 0 \qquad \text{whenever } m > n, \tag{4.9}$$

and so (4.9) and Definition 1.(iv) together give (4.8).

Now take inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$, and consider $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$, where we define

$$\lambda_{n;m} = 0 \qquad \text{whenever } m > n. \tag{4.10}$$

Then Definition 1.(i) follows directly from (4.10) by taking $n = 0$. Also, Definition 1.(ii) follows from Definition 12.(i) since $M \leq N$. Next, Definition 12.(ii) gives

$$\lambda_{n;m+1} \leq \lambda_{n-1;m} \leq \lambda_{n;m} \qquad \forall n = 2, \ldots, N, \; m = 1, \ldots, n-1. \tag{4.11}$$

Using the nonnegativity of $\{\lambda_n\}_{n=1}^{N}$ along with Definition 12.(i) and an iterative application of the left-hand inequality of (4.11) then gives

$$0 \leq \lambda_N = \lambda_{N;N} \leq \cdots \leq \lambda_{n;n} \qquad \forall n = 1, \ldots, N.$$

Combining this with an iterative application of the right-hand inequality of (4.11) then gives

$$0 \leq \lambda_{m;m} \leq \cdots \leq \lambda_{n;m} \qquad \forall n \geq m. \tag{4.12}$$

For Definition 1.(iii), we need to show (4.6) and (4.7) for every $n = 1, \ldots, N$. Considering (4.10), when $n = 1$, (4.6) and (4.7) together become $\lambda_{1;1} \geq 0$, which follows from (4.12). Also, when $n > M$, (4.11) immediately gives (4.6), while (4.7) follows from both (4.12) and (4.11):

$$0 \leq \lambda_{n;M+1} \leq \lambda_{n-1;M} \leq \lambda_{n;M}.$$

For the case $2 \leq n \leq M$, note that (4.11) gives the inequalities in (4.6) whenever $m \leq n-1$. Furthermore when $m = n$, (4.10) gives $\lambda_{n;n+1} = \lambda_{n-1;n} = 0$, and so the inequalities in (4.6) become $\lambda_{n;n} \geq 0$, which follows from (4.12). Otherwise when $m > n$, the inequalities in (4.6) become $0 \leq 0 \leq 0$ by (4.10). To finish the case $2 \leq n \leq M$, we need to prove (4.7). When $n = M$, (4.7) becomes $\lambda_{n;n} \geq 0$, which follows from (4.12). Otherwise when $n < M$, (4.7) becomes $0 \leq 0 \leq 0$ by (4.10).

At this point, all that remains to be shown is that $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ satisfies Definition 1.(iv). For $n \leq M$, (4.10) and Definition 12.(iii) together imply

$$\sum_{m=1}^{M} \lambda_{n;m} = \sum_{m=1}^{n} \lambda_{n;m} = \sum_{m=1}^{n} \mu_m. \tag{4.13}$$

Next, note that (4.12), Definition 12.(i), and our assumption that $\lambda_n = 0$ for every $n > M$ gives

$$0 \le \lambda_{n;m} \le \lambda_{N;m} = \lambda_m = 0 \qquad \text{whenever } n \ge m > M.$$

Thus, $\lambda_{n;m} = 0$ whenever $n \ge m > M$; when $n > M$, we can combine this with Definition 12.(iii) to get (4.13). □

This result, when coupled with a complete constructive characterization of all valid inner eigensteps provides a systematic method for constructing any and all valid outer eigensteps, thereby making Step A of Theorem 2 explicit.

4.2 Top Kill and the existence of eigensteps

In Chapter 3, Theorem 7 gave us an explicit construction of all sequences of vectors whose partial-frame-operator spectra match the eigensteps chosen in Step A of Theorem 2. We now focus on the problem of performing Step A explicitly, namely choosing a sequence of outer eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ which satisfy Definition 1. By Theorem 14, we see that every sequence of outer eigensteps corresponds to a unique sequence of inner eigensteps. Note that if a sequence of inner eigensteps exists, then $\{\lambda_n\}_{n=1}^{N}$ necessarily majorizes $\{\mu_n\}_{n=1}^{N}$ since

$$\sum_{m=1}^{n} \lambda_m \ge \sum_{m=1}^{n} \lambda_{n;m} = \sum_{m=1}^{n} \mu_m.$$

In this section, we prove the converse, specifically that if $\{\lambda_n\}_{n=1}^{N}$ majorizes $\{\mu_n\}_{n=1}^{N}$, then a corresponding sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$ exists. To be clear, the Schur-Horn Theorem gives that this set is nonempty if and only if $\{\lambda_n\}_{n=1}^{N}$ majorizes $\{\mu_n\}_{n=1}^{N}$. Horn and Johnson [33] already proved that such a sequence exists, however, their proof provides no intuition as to how to explicitly construct such an interlacing sequence. The main algorithm of this section, *Top Kill*, not only provides an alternative proof of Horn and Johnson's approach, but it refines Step A of Theorem 2 by giving an explicit construction of

a feasible set of eigensteps. In the next section, we use these results to further parametrize the set of all eigensteps. We now motivate the Top Kill algorithm with an example:

Example 15. Let $N = 3$, $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$ and $\{\mu_1, \mu_2, \mu_3\} = \{1, 1, 1\}$. Since this spectrum majorizes these lengths, we claim that there exists a corresponding sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{3}$. That is, recalling Definition 12, we claim it is possible to find values $\{\lambda_{1;1}\}$ and $\{\lambda_{2;1}, \lambda_{2;2}\}$ which satisfy the interlacing requirements (ii) that $\{\lambda_{1;1}\} \sqsubseteq \{\lambda_{2;1}, \lambda_{2;2}\} \sqsubseteq \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$ as well as the trace requirements (iii) that $\lambda_{1;1} = 1$ and $\lambda_{2;1} + \lambda_{2;2} = 2$. Indeed, every such sequence of eigensteps is given by the table:

$$\begin{array}{cccc} n & 1 & 2 & 3 \\ \hline \lambda_{n;3} & & & \frac{1}{2} \\ \lambda_{n;2} & & x & \frac{3}{4} \\ \lambda_{n;1} & 1 & 2-x & \frac{7}{4} \end{array} \tag{4.14}$$

where x is required to satisfy

$$\tfrac{1}{2} \leq x \leq \tfrac{3}{4} \leq 2 - x \leq \tfrac{7}{4}, \qquad x \leq 1 \leq 2 - x. \tag{4.15}$$

Clearly, any $x \in [\frac{1}{2}, \frac{3}{4}]$ will do. However, when N is large, the table analogous to (4.14) will contain many more variables, leading to a system of inequalities which is much larger and more complicated than (4.15). In such settings, it is not obvious how to construct even a single valid sequence of eigensteps. As such, we consider this same simple example from a different perspective—one that leads to an eigenstep construction algorithm which is easily implementable regardless of the size of N.

The key idea is to view the task of constructing eigensteps as iteratively building a staircase in which the nth level is λ_n units long. For this example in particular, our goal is to build a three-step staircase where the bottom level has length $\frac{7}{4}$, the second level has length $\frac{3}{4}$, and the top level has length $\frac{1}{2}$; the profile of such a staircase is outlined in black in each

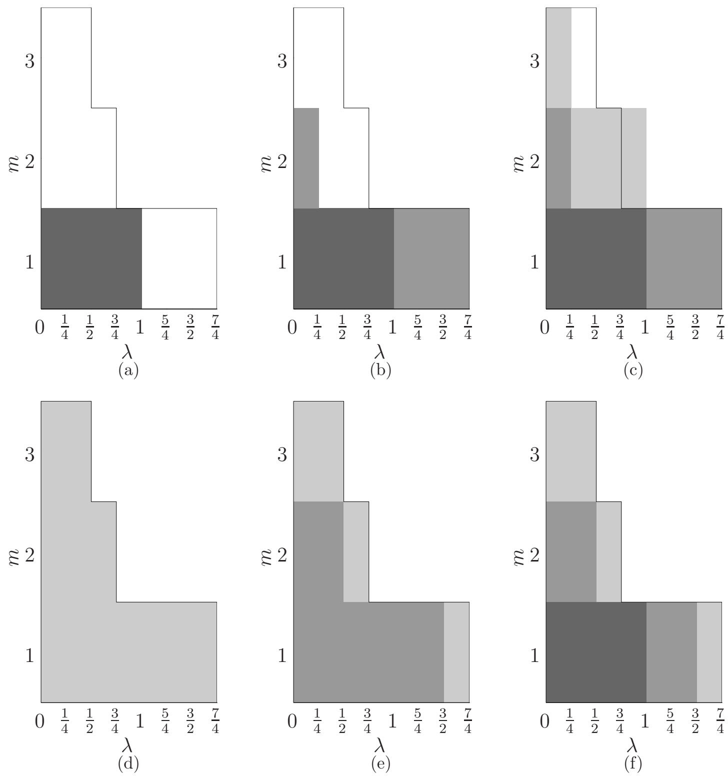

Figure 4.1: Two attempts at iteratively building a sequence of inner eigensteps for $\{\lambda_1,\lambda_2,\lambda_3\} = \{\frac{7}{4},\frac{3}{4},\frac{1}{2}\}$ and $\{\mu_1,\mu_2,\mu_3\} = \{1,1,1\}$. As detailed in Example 15, the first row represents a failed attempt in which we greedily complete the first level before focusing on those above it. The failure arises from a lack of foresight: the second step does not build sufficient foundation for the third. The second row represents a second attempt, one that is successful. There, we begin with the final desired staircase and work backwards. That is, we chip away at the three-level staircase (d) to produce a two-level one (e), and then chip away at it to produce a one-level one (f). In each step, we do this by removing as much as possible from the top level before turning our attention to the lower levels, subject to the interlacing constraints. We refer to this algorithm for iteratively producing $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$ from $\{\lambda_{n;m}\}_{m=1}^{n}$ as Top Kill. Theorem 16 shows that Top Kill will always produce a valid sequence of eigensteps from any desired spectrum $\{\lambda_n\}_{n=1}^{N}$ that majorizes a given desired sequence of lengths $\{\mu_n\}_{n=1}^{N}$.

of the six subfigures of Figure 4.1. The benefit of visualizing eigensteps in this way is that the interlacing and trace conditions become intuitive staircase-building rules. Specifically, up until the nth step, we will have built a staircase whose levels are $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$. To build on top of this staircase, we use n blocks of height 1 whose areas sum to μ_n. Each of these

n new blocks is added to its corresponding level of the current staircase, and is required to rest entirely on top of what has been previously built. This requirement corresponds to the interlacing condition (ii) of Definition 12, while the trace condition (iii) corresponds to the fact that the block areas sum to μ_n.

This intuition in mind, we now try to build such a staircase from the ground up. In the first step (Figure 4.1(a)), we are required to place a single block of area $\mu_1 = 1$ on the first level. The length of this first level is $\lambda_{1;1} = \mu_1$. In the second step, we build up and out from this initial block, placing two new blocks—one on the first level and another on the second—whose total area is $\mu_2 = 1$. The lengths $\lambda_{2;1}$ and $\lambda_{2;2}$ of the new first and second levels depends on how these two blocks are chosen. In particular, choosing first and second level blocks of area $\frac{3}{4}$ and $\frac{1}{4}$, respectively, results in $\{\lambda_{2;1}, \lambda_{2;2}\} = \{\frac{7}{4}, \frac{1}{4}\}$ (Figure 4.1(b)); this corresponds to a greedy pursuit of the final desired spectrum $\{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$, fully completing the first level before turning our attention to the second. The problem with this greedy approach is that it doesn't always work, as this example illustrates. Indeed, in the third and final step, we build up and out from the staircase of Figure 4.1(b) by adding three new blocks—one each for the first, second and third levels—whose total area is $\mu_3 = 1$. However, in order to maintain interlacing, the new top block must rest entirely on the existing second level, meaning that its length $\lambda_{3;3} \leq \lambda_{2;2} = \frac{1}{4}$ cannot equal the desired value of $\frac{1}{2}$. That is, because of our poor choice in the second step, the "best" we can now do is $\{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} = \{\frac{7}{4}, 1, \frac{1}{4}\}$ (Figure 4.1(c)):

n	1	2	3
$\lambda_{n;3}$			$\frac{1}{4}$
$\lambda_{n;2}$		$\frac{1}{4}$	1
$\lambda_{n;1}$	1	$\frac{7}{4}$	$\frac{7}{4}$

The reason this greedy approach fails is that it doesn't plan ahead. Indeed, it treats the bottom levels of the staircase as the priority when, in fact, the opposite is true: the top

levels are the priority since they require the most foresight. In particular, for $\lambda_{3;3}$ to achieve its desired value of $\frac{1}{2}$ in the third step, one must lay a suitable foundation in which $\lambda_{2;2} \geq \frac{1}{2}$ in the second step.

In light of this realization, we make another attempt at building our staircase. This time we begin with the final desired spectrum $\{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$ (Figure 4.1(d)) and work backwards. From this perspective, our task now is to remove three blocks—the entirety of the top level, and portions of the first and second levels—whose total area is $\mu_3 = 1$. Here, the interlacing requirement translates to only being permitted to remove portions of the staircase that were already exposed to the surface at the end of the previous step. After lopping off the top level, which has area $\lambda_{3;3} = \frac{1}{2}$, we need to decide how to chip away $\mu_1 - \lambda_{3;3} = 1 - \frac{1}{2} = \frac{1}{2}$ units of area from the first and second levels, subject to this constraint. At this point, we observe that in the step that follows, our first task will be to remove the remaining portion of the second level. As such, it is to our advantage to remove as much of the second level as possible in the current step, and only then turn our attention to the lower levels. That is, we follow Thomas Jefferson's adage, "Never put off until tomorrow what you can do today." We dub this approach Top Kill since it "kills" off as much as possible from the top portions of the staircase. For this example in particular, interlacing implies that we can at most remove a block of area $\frac{1}{4}$ from the second level, leaving $\frac{1}{4}$ units of area to be removed from the first; the resulting two-level staircase—the darker shade in Figure 4.1(e)—has levels of lengths $\{\lambda_{2;1}, \lambda_{2;2}\} = \{\frac{3}{2}, \frac{1}{2}\}$. In the second step, we then apply this same philosophy, removing the entire second level and a block of area $\mu_2 - \lambda_{2;2} = 1 - \frac{1}{2} = \frac{1}{2}$ from the first, resulting in the one-level staircase (Figure 4.1(f)) in which $\{\lambda_{1;1}\} = 1$. That is, by working backwards we have produced a valid sequence of eigensteps:

n	1	2	3
$\lambda_{n;3}$			$\frac{1}{4}$
$\lambda_{n;2}$		$\frac{1}{2}$	$\frac{3}{4}$
$\lambda_{n;1}$	1	$\frac{3}{2}$	$\frac{7}{4}$

The preceding example illustrated a systematic "Top Kill" approach for building eigensteps; we now express these ideas more rigorously. As can be seen in the bottom row of Figure 4.1, Top Kill generally picks $\lambda_{n-1;m} := \lambda_{n;m+1}$ for the larger m's. Top Kill also picks $\lambda_{n-1;m} := \lambda_{n;m}$ for the smaller m's. The level that separates the larger m's from the smaller m's is the lowest level from which a nontrivial area is removed. For this level, say level k, we have $\lambda_{n;k+1} < \mu_n \leq \lambda_{n;k}$. In the levels above k, we have already removed a total of $\lambda_{n;k+1}$ units of area, leaving $\mu_n - \lambda_{n;k+1}$ to be chipped away from $\lambda_{n;k}$, yielding $\lambda_{n-1;k} := \lambda_{n;k} - (\mu_n - \lambda_{n;k+1})$. The following theorem confirms that Top Kill always produces eigensteps whenever it is possible to do so:

Theorem 16. *Suppose $\{\lambda_{n;m}\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$, and define $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$ according to Top Kill, that is, pick any k such that $\lambda_{n;k+1} \leq \mu_n \leq \lambda_{n;k}$, and for each $m = 1, \ldots, n-1$, define:*

$$\lambda_{n-1;m} := \begin{cases} \lambda_{n;m}, & 1 \leq m \leq k-1, \\ \lambda_{n;k} + \lambda_{n;k+1} - \mu_n, & m = k, \\ \lambda_{n;m+1}, & k+1 \leq m \leq n-1. \end{cases} \tag{4.16}$$

Then $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^{n}$ and $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$.

Furthermore, given any nonnegative nonincreasing sequences $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$ such that $\{\lambda_n\}_{n=1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$, define $\lambda_{N;m} := \lambda_m$ for every $m = 1, \ldots, N$, and for each $n = N, \ldots, 2$, consecutively define $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$ according to Top Kill. Then $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$ are inner eigensteps.

Proof. For the sake of notational simplicity, we denote $\{\alpha_m\}_{m=1}^{n-1} := \{\lambda_{n-1;m}\}_{m=1}^{n-1}$ and $\{\beta_m\}_{m=1}^{n} := \{\lambda_{n;m}\}_{m=1}^{n}$. Since $\{\beta_m\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$, we necessarily have that $\beta_n \leq \mu_n \leq \mu_1 \leq \beta_1$

and so there exists $k = 1, \ldots, n-1$ such that $\beta_{k+1} \leq \mu_n \leq \beta_k$. Though this k may not be unique when subsequent β_m's are equal, a quick inspection reveals that any appropriate choice of k will yield the same α_m's, and so Top Kill is well-defined. To prove $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^{n}$, we need to show that:

$$\beta_{m+1} \leq \alpha_m \leq \beta_m \tag{4.17}$$

for every $m = 1, \ldots, n-1$. If $1 \leq m \leq k-1$, then $\alpha_m := \beta_m$, and so the right-hand inequality of (4.17) holds with equality, at which point the left-hand inequality is immediate. Similarly, if $k+1 \leq m \leq n-1$, then $\alpha_m := \beta_{m+1}$, and so (4.17) holds with equality on the left-hand side. Lastly if $m = k$, then $\alpha_k := \beta_k + \beta_{k+1} - \mu_n$, and our assumption that $\beta_{k+1} \leq \mu_n \leq \beta_k$ gives (4.17) in this case:

$$\beta_{k+1} \leq \beta_k + \beta_{k+1} - \mu_n \leq \beta_k.$$

Thus, Top Kill produces $\{\alpha_m\}_{m=1}^{n-1}$ such that $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^{n}$. We next show that $\{\alpha_m\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$. If $j \leq k-1$, then since $\{\beta_m\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$, we have:

$$\sum_{m=1}^{j} \alpha_m = \sum_{m=1}^{j} \beta_m \geq \sum_{m=1}^{j} \mu_m.$$

On the other hand, if $j \geq k$, we have:

$$\sum_{m=1}^{j} \alpha_m = \sum_{m=1}^{k-1} \beta_m + (\beta_k + \beta_{k+1} - \mu_n) + \sum_{m=k+1}^{j} \beta_{m+1} = \sum_{m=1}^{j+1} \beta_m - \mu_n, \tag{4.18}$$

with the understanding that a sum over an empty set of indices is zero. We continue (4.18) by using the facts that $\{\beta_m\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$ and $\mu_{j+1} \geq \mu_n$:

$$\sum_{m=1}^{j} \alpha_m = \sum_{m=1}^{j+1} \beta_m - \mu_n \geq \sum_{m=1}^{j+1} \mu_m - \mu_n \geq \sum_{m=1}^{j} \mu_m. \tag{4.19}$$

Note that when $j = n$, the inequalities in (4.19) become equalities, giving the final trace condition.

For the final conclusion, note that one application of Top Kill transforms a sequence $\{\lambda_{n;m}\}_{m=1}^{n}$ that majorizes $\{\mu_m\}_{m=1}^{n}$ into a shorter sequence $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$ that interlaces with

$\{\lambda_{n;m}\}_{m=1}^{n}$ and majorizes $\{\mu_m\}_{m=1}^{n-1}$. As such, one may indeed start with $\lambda_{N;m} := \lambda_m$ and apply Top Kill $N-1$ times to produce a sequence $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$ that immediately satisfies Definition 12. $\square$

4.3 Parametrizing eigensteps

In the previous section, we discussed Top Kill, an algorithm designed to construct a sequence of inner eigensteps from given nonnegative nonincreasing sequences $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$. As previously mentioned, the Schur-Horn Theorem gives that the set of all inner eigensteps is nonempty if and only if $\{\lambda_n\}_{n=1}^{N}$ majorizes $\{\mu_n\}_{n=1}^{N}$. Indeed, as noted at the beginning of the previous section, if such a sequence of eigensteps exists then we necessarily have that $\{\lambda_n\}_{n=1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$. Conversely, if $\{\lambda_n\}_{n=1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$ then Theorem 16 states that Top Kill will produce a valid sequence of eigensteps from $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$. That is, the results of the previous section give an alternative proof of the Schur-Horn Theorem.

In this section, we use the intuition underlying Top Kill to find a systematic method for producing all such eigensteps. If the values $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N-1}$ are treated as independent variables, it can easily be shown that the set of all inner eigensteps for a given $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$ form a convex polytope in $\mathbb{R}^{n(n-1)/2}$. The main result of this section gives a complete characterization of the set of all eigensteps for a given $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$. Unlike Top Kill which gives just one strategy for constructing a sequence of eigensteps required by Step A of Theorem 2, the results of this section provide a systematic method for producing any feasible sequence of eigensteps. In the work that follows, we view these non-Top-Kill-produced eigensteps as the result of applying generalizations of Top Kill to $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$.

Recall Example 13 at the beginning of this Chapter where $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} = \{\frac{5}{3}, \frac{5}{3}, \frac{5}{3}, 0, 0\}$ and $\mu_n = 1$ for all $n = 1, \dots, 5$. In order for (4.1) to be a valid sequence of eigensteps, the 10 unknown values in (4.1) must satisfy the interlacing and trace conditions

(ii) and (iii) of Definition 12. The set of all such eigensteps form a convex polytope in $\mathbb{R}^{10}$. Taking advantage of the interlacing and trace conditions, we reduce these 10 unknowns to just two unknowns in (4.3). The variables x and y must then be chosen so that they satisfy the system of inequalities given in (4.4). In general, for any $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$, constructing all sequences of eigensteps will require simplifying a large system of inequalities which motivates us to find a more systematic way of parametrizing the resulting convex polytope.

In this section, we suggest a different method for parametrizing the polytope: to systematically pick the values $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{4}$ one at a time. Top Kill is one way to do this: working from the top levels down, we chip away $\mu_5 = 1$ units of area from $\{\lambda_{5;m}\}_{m=1}^{5}$ to successively produce $\lambda_{4;4} = 0$, $\lambda_{4;3} = \frac{2}{3}$, $\lambda_{4;2} = \frac{5}{3}$ and $\lambda_{4;1} = \frac{5}{3}$; we then repeat this process to transform $\{\lambda_{4;m}\}_{m=1}^{4}$ into $\{\lambda_{3;m}\}_{m=1}^{3}$, and so on; the specific values can be obtained by letting $(x, y) = (0, \frac{1}{3})$ in (4.3). We seek to generalize Top Kill to find *all* ways of picking the $\lambda_{n;m}$'s one at a time. As in Top Kill, we work backwards: we first find all possibilities for $\lambda_{4;4}$, then the possibilities for $\lambda_{4;3}$ in terms of our choice of $\lambda_{4;4}$, then the possibilities for $\lambda_{4;2}$ in terms of our choices of $\lambda_{4;4}$ and $\lambda_{4;3}$, and so on. That is, we iteratively parametrize our convex polytope in the following order:

$$\lambda_{4;4}, \quad \lambda_{4;3}, \quad \lambda_{4;2}, \quad \lambda_{4;1}, \quad \lambda_{3;3}, \quad \lambda_{3;2}, \quad \lambda_{3;1}, \quad \lambda_{2;2}, \quad \lambda_{2;1}, \quad \lambda_{1;1}.$$

More generally, for any $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$ such that $\{\lambda_n\}_{n=1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$ we construct every possible sequence of eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{n}\}_{n=1}^{N}$ by finding all possibilities for any given $\lambda_{n-1;k}$ in terms of $\lambda_{n';m}$ where either $n - 1 < n'$ or $n - 1 = n'$ and $k < m$. Certainly, any permissible choice for $\lambda_{n-1;k}$ must satisfy the interlacing criteria (ii) of Definition 12, and so we have bounds $\lambda_{n;k+1} \leq \lambda_{n-1;k} \leq \lambda_{n;k}$. Other necessary bounds arise from the majorization conditions. Indeed, in order to have both $\{\lambda_{n;m}\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$ and $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$ we need:

$$\mu_n = \sum_{m=1}^{n} \mu_m - \sum_{m=1}^{n-1} \mu_m = \sum_{m=1}^{n} \lambda_{n;m} - \sum_{m=1}^{n-1} \lambda_{n-1;m}, \tag{4.20}$$

and so we may view μ_n as the total change between the eigenstep spectra. Having already selected $\lambda_{n-1;n-1},\dots,\lambda_{n-1;k+1}$, we've already imposed a certain amount of change between the spectra, and so we are limited in how much we can change the kth eigenvalue. Continuing (4.20), this fact can be expressed as:

$$\mu_n = \lambda_{n;n} + \sum_{m=1}^{n-1}(\lambda_{n;m} - \lambda_{n-1;m}) \geq \lambda_{n;n} + \sum_{m=k}^{n-1}(\lambda_{n;m} - \lambda_{n-1;m}), \tag{4.21}$$

where the inequality follows from the fact that the summands $\lambda_{n;m} - \lambda_{n-1;m}$ are nonnegative if $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$ is to be chosen so that $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^{n}$. Rearranging (4.21) then gives a second lower bound on $\lambda_{n-1;k}$ to go along with our previously mentioned requirement that $\lambda_{n;k+1} \leq \lambda_{n-1;k}$:

$$\sum_{m=k}^{n} \lambda_{n;m} - \sum_{m=k+1}^{n-1} \lambda_{n-1;m} - \mu_n \leq \lambda_{n-1;k}.$$

That is, we necessarily have that

$$\max\left\{\lambda_{n;k+1}, \sum_{m=k}^{n} \lambda_{n;m} - \sum_{m=k+1}^{n-1} \lambda_{n-1;m} - \mu_n\right\} \leq \lambda_{n-1;k}. \tag{4.22}$$

We next apply the intuition behind Top Kill to obtain other upper bounds on $\lambda_{n-1;k}$ to go along with our previously mentioned requirement that $\lambda_{n-1;k} \leq \lambda_{n;k}$. Recall that we have already selected $\{\lambda_{n;m}\}_{m=k+1}^{n-1}$ and are attempting to find all possible choices $\lambda_{n-1;k}$ that will allow the remaining values $\{\lambda_{n-1;m}\}_{m=1}^{k-1}$ to be chosen in such a way that:

$$\{\lambda_{n-1;m}\}_{m=1}^{n-1} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^{n}, \qquad \{\lambda_{n-1;m}\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}. \tag{4.23}$$

To do this, we recall our staircase-building intuition from the previous section: if it is possible to build a given staircase, then one way to do this is to assign maximal priority to the highest levels, as these are the most difficult to build. As such, for a given choice of $\lambda_{n-1;k}$, if it is possible to choose $\{\lambda_{n-1;m}\}_{m=1}^{k-1}$ in such a way that (4.23) holds, then it is reasonable to expect that one way of doing this is to pick $\lambda_{n-1;k-1}$ by chipping away as much as possible from $\lambda_{n;k-1}$, then pick $\lambda_{n-1;k-2}$ by chipping away as much as possible from $\lambda_{n;k-2}$, and so on. That is, we pick some arbitrary value $\lambda_{n-1;k}$ and to test its legitimacy,

apply the Top Kill algorithm to construct the remaining undetermined values $\{\lambda_{n-1;m}\}_{m=1}^{k-1}$; we then check whether or not $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^{n}$.

To be precise, note that prior to applying Top Kill, the remaining spectrum is $\{\lambda_{n;m}\}_{m=1}^{k-1}$, and that the total amount we will chip away from this spectrum is:

$$\mu_n - \Big(\lambda_{n;n} + \sum_{m=k}^{n-1}(\lambda_{n;m} - \lambda_{n-1;m})\Big). \tag{4.24}$$

To ensure that our choice of $\lambda_{n-1;k-1}$ satisfies $\lambda_{n;k} \le \lambda_{n-1;k-1}$, we artificially reintroduce $\lambda_{n;k}$ to both (4.24) and the remaining spectrum $\{\lambda_{n;m}\}_{m=1}^{k-1}$ before applying Top Kill. That is, we apply Top Kill to $\{\beta_m\}_{m=1}^{n} := \{\lambda_{n;m}\}_{m=1}^{k} \cup \{0\}_{m=k+1}^{n}$ where:

$$\mu := \mu_n - \Big(\lambda_{n;n} + \sum_{m=k}^{n-1}(\lambda_{n;m} - \lambda_{n-1;m})\Big) + \lambda_{n;k} = \mu_n - \sum_{m=k+1}^{n} \lambda_{n;m} + \sum_{m=k}^{n-1} \lambda_{n-1;m}. \tag{4.25}$$

Specifically in light of Theorem 16, in order to optimally subtract μ units of area from $\{\beta_m\}_{m=1}^{n}$, we first pick j such that $\beta_{j+1} \le \mu \le \beta_j$. We then use (4.16) to produce a zero-padded version of the remaining new spectrum $\{\lambda_{n-1;m}\}_{m=1}^{k-1} \cup \{0\}_{m=k}^{n}$:

$$\lambda_{n-1;m} = \begin{cases} \lambda_{n;m}, & 1 \le m \le j-1, \\ \lambda_{n;j} + \lambda_{n;j+1} - \mu_n + \displaystyle\sum_{m'=k+1}^{n} \lambda_{n;m'} - \sum_{m'=k}^{n-1} \lambda_{n-1;m'}, & m = j \\ \lambda_{n;m+1}, & j+1 \le m \le k-1. \end{cases}$$

Picking l such that $j+1 \le l \le k$, we now sum the above values of $\lambda_{n-1;m}$ to obtain:

$$\begin{aligned} \sum_{m=1}^{l-1} \lambda_{n-1;m} &= \sum_{m=1}^{j-1} \lambda_{n-1;m} + \lambda_{n-1;j} + \sum_{m=j+1}^{l-1} \lambda_{n-1;m} \\ &= \sum_{m=1}^{l} \lambda_{n;m} - \mu_n + \sum_{m=k+1}^{n} \lambda_{n;m} - \sum_{m=k}^{n-1} \lambda_{n-1;m}. \end{aligned} \tag{4.26}$$

Adding $\displaystyle\sum_{m=1}^{n} \mu_m - \sum_{m=1}^{n} \lambda_{n;m} = 0$ to the right-hand side of (4.26) then yields:

$$\begin{aligned} \sum_{m=1}^{l-1} \lambda_{n-1;m} &= \sum_{m=1}^{l} \lambda_{n;m} - \mu_n + \sum_{m=k+1}^{n} \lambda_{n;m} - \sum_{m=k}^{n-1} \lambda_{n-1;m} + \sum_{m=1}^{n} \mu_m - \sum_{m=1}^{n} \lambda_{n;m} \\ &= \sum_{m=1}^{n-1} \mu_m - \sum_{m=l+1}^{k} \lambda_{n;m} - \sum_{m=k}^{n-1} \lambda_{n-1;m}. \end{aligned} \tag{4.27}$$

Now, in order for $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$ as desired, (4.27) must satisfy:

$$\sum_{m=1}^{l-1}\mu_m \le \sum_{m=1}^{l-1}\lambda_{n-1;m} = \sum_{m=1}^{n-1}\mu_m - \sum_{m=l+1}^{k}\lambda_{n;m} - \sum_{m=k}^{n-1}\lambda_{n-1;m}. \tag{4.28}$$

Solving for $\lambda_{n-1;k}$ in (4.28) then gives:

$$\lambda_{n-1;k} \le \sum_{m=l}^{n-1}\mu_m - \sum_{m=l+1}^{k}\lambda_{n;m} - \sum_{m=k+1}^{n-1}\lambda_{n-1;m}. \tag{4.29}$$

Note that, according to how we derived it, (4.29) is valid when $j+1 \le l \le k$. Combining the interlacing requirement that $\lambda_{n-1;k} \le \lambda_{n;k}$ with (4.29), we have

$$\lambda_{n-1;k} \le \min\left\{\lambda_{n;k}, \min_{l=j+1,\ldots,k}\left\{\sum_{m=l}^{n-1}\mu_m - \sum_{m=l+1}^{k}\lambda_{n;m} - \sum_{m=k+1}^{n-1}\lambda_{n-1;m}\right\}\right\}. \tag{4.30}$$

As established in the following theorem, this bound actually holds when $l = 1,\ldots,k$. Overall, (4.22) and (4.30) are precisely the bounds that we verify in the following result:

Theorem 17. *Suppose $\{\lambda_{n;m}\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$. Then $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$ and $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^{n}$ if and only if $\lambda_{n-1;k} \in [A_{n-1;k}, B_{n-1;k}]$ for every $k = 1,\ldots,n-1$, where:*

$$A_{n-1;k} := \max\left\{\lambda_{n;k+1}, \sum_{m=k}^{n}\lambda_{n;m} - \sum_{m=k+1}^{n-1}\lambda_{n-1;m} - \mu_n\right\}, \tag{4.31}$$

$$B_{n-1;k} := \min\left\{\lambda_{n;k}, \min_{l=1,\ldots,k}\left\{\sum_{m=l}^{n-1}\mu_m - \sum_{m=l+1}^{k}\lambda_{n;m} - \sum_{m=k+1}^{n-1}\lambda_{n-1;m}\right\}\right\}. \tag{4.32}$$

Here, we use the convention that sums over empty sets of indices are zero. Moreover, suppose $\lambda_{n-1;n-1},\ldots,\lambda_{n-1;k+1}$ are consecutively chosen to satisfy these bounds. Then $A_{n-1;k} \le B_{n-1;k}$, and so $\lambda_{n-1;k}$ can also be chosen from such an interval.

Proof. For the sake of notational simplicity, we let $\{\alpha_m\}_{m=1}^{n-1} := \{\lambda_{n-1;m}\}_{m=1}^{n-1}$, $\{\beta_m\}_{m=1}^{n} := \{\lambda_{n;m}\}_{m=1}^{n}$, $A_k := A_{n-1;k}$, and $B_k := B_{n-1;k}$.

($\Rightarrow$) Suppose $\{\alpha_m\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$ and $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^{n}$. Fix any particular $k = 1,\ldots,n-1$. Note that interlacing gives $\beta_{k+1} \le \alpha_k \le \beta_k$, which accounts for the first entries in

(4.31) and (4.32). We first show $\alpha_k \geq A_k$. Since $\{\beta_m\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$ and $\{\alpha_m\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$, then:

$$\mu_n = \sum_{m=1}^{n} \mu_m - \sum_{m=1}^{n-1} \mu_m = \sum_{m=1}^{n} \beta_m - \sum_{m=1}^{n-1} \alpha_m = \beta_n + \sum_{m=1}^{n-1} (\beta_m - \alpha_m). \tag{4.33}$$

Since $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^{n}$, the summands in (4.33) are nonnegative, and so:

$$\mu_n \geq \beta_n + \sum_{m=k}^{n-1} (\beta_m - \alpha_m) = \sum_{m=k}^{n} \beta_m - \sum_{m=k+1}^{n-1} \alpha_m - \alpha_k. \tag{4.34}$$

Isolating α_k in (4.34) and combining with the fact that $\alpha_k \geq \beta_{k+1}$ gives $\alpha_k \geq A_k$. We next show that $\alpha_k \leq B_k$. Fix $l = 1, \dots, k$. Then $\{\alpha_m\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$ implies $\sum_{m=1}^{l-1} \alpha_m \geq \sum_{m=1}^{l-1} \mu_m$ and $\sum_{m=1}^{n-1} \alpha_m = \sum_{m=1}^{n-1} \mu_m$, and so subtracting gives:

$$\sum_{m=l}^{n-1} \mu_m \geq \sum_{m=l}^{n-1} \alpha_m = \sum_{m=k}^{n-1} \alpha_m + \sum_{m=l}^{k-1} \alpha_m \geq \sum_{m=k}^{n-1} \alpha_m + \sum_{m=l}^{k-1} \beta_{m+1}, \tag{4.35}$$

where the second inequality follows from $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^{n}$. Since our choice for $l = 1, \dots, k$ was arbitrary, isolating α_k in (4.35) and combining with the fact that $\alpha_k \leq \beta_k$ gives $\alpha_k \leq B_k$.

($\Leftarrow$) Now suppose $A_k \leq \alpha_k \leq B_k$ for every $k = 1, \dots, n-1$. Then the first entries in (4.31) and (4.32) give $\beta_{k+1} \leq \alpha_k \leq \beta_k$ for every $k = 1, \dots, n-1$, that is, $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^{n}$. It remains to be shown that $\{\alpha_m\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$. Since $\alpha_k \leq B_k$ for every $k = 1, \dots, n-1$, then:

$$\alpha_k \leq \sum_{m=l}^{n-1} \mu_m - \sum_{m=l+1}^{k} \beta_m - \sum_{m=k+1}^{n-1} \alpha_m \qquad \forall k = 1, \dots, n-1, \;\; l = 1, \dots, k. \tag{4.36}$$

Rearranging (4.36) in the case where $l = k$ gives:

$$\sum_{m=k}^{n-1} \alpha_m \leq \sum_{m=k}^{n-1} \mu_m \qquad \forall k = 1, \dots, n-1. \tag{4.37}$$

Moreover, $\alpha_1 \geq A_1$ implies $\alpha_1 \geq \sum_{m=1}^{n} \beta_m - \sum_{m=2}^{n-1} \alpha_m - \mu_n$. Rearranging this inequality and applying $\{\beta_m\}_{m=1}^{n} \succeq \{\mu_m\}_{m=1}^{n}$ then gives:

$$\sum_{m=1}^{n-1} \alpha_m \geq \sum_{m=1}^{n} \beta_m - \mu_n = \sum_{m=1}^{n-1} \mu_m. \tag{4.38}$$

Combining (4.38) with (4.37) in the case where $k = 1$ gives:

$$\sum_{m=1}^{n-1} \alpha_m = \sum_{m=1}^{n-1} \mu_m. \tag{4.39}$$

Subtracting (4.37) from (4.39) completes the proof that $\{\alpha_m\}_{m=1}^{n-1} \geq \{\mu_m\}_{m=1}^{n-1}$.

For the final claim, we first show that the claim holds for $k = n-1$, namely that $A_{n-1} \leq B_{n-1}$. Explicitly, we need to show:

$$\max\{\beta_n, \beta_{n-1} + \beta_n - \mu_n\} \leq \min\left\{\beta_{n-1}, \min_{l=1,\ldots,n-1}\left\{\sum_{m=l}^{n-1} \mu_m - \sum_{m=l+1}^{n-1} \beta_m\right\}\right\}. \tag{4.40}$$

Note that (4.40) is equivalent to the following inequalities holding simultaneously:

(i) $\beta_n \leq \beta_{n-1}$,

(ii) $\beta_{n-1} + \beta_n - \mu_n \leq \beta_{n-1}$,

(iii) $\beta_n \leq \sum_{m=l}^{n-1} \mu_m - \sum_{m=l+1}^{n-1} \beta_m \quad \forall l = 1, \ldots, n-1$,

(iv) $\beta_{n-1} + \beta_n - \mu_n \leq \sum_{m=l}^{n-1} \mu_m - \sum_{m=l+1}^{n-1} \beta_m \quad \forall l = 1, \ldots, n-1$.

First, (i) follows immediately from the fact that $\{\beta_m\}_{m=1}^{n}$ is nonincreasing. Next, rearranging (ii) gives $\beta_n \leq \mu_n$, which follows from $\{\beta_m\}_{m=1}^{n} \geq \{\mu_m\}_{m=1}^{n}$. For (iii), the facts that $\{\beta_m\}_{m=1}^{n} \geq \{\mu_m\}_{m=1}^{n}$ and $\{\mu_m\}_{m=1}^{n}$ is nonincreasing imply:

$$\sum_{m=l+1}^{n} \beta_m \leq \sum_{m=l+1}^{n} \mu_m \leq \sum_{m=l}^{n-1} \mu_m \qquad \forall l = 1, \ldots, n-1,$$

which in turn implies (iii). Also for (iv), the facts that $\{\beta_m\}_{m=1}^{n}$ is nonincreasing and $\{\beta_m\}_{m=1}^{n} \geq \{\mu_m\}_{m=1}^{n}$ imply:

$$\beta_{n-1} + \sum_{m=l+1}^{n} \beta_m \leq \sum_{m=l}^{n} \beta_m \leq \sum_{m=l}^{n} \mu_m \qquad \forall l = 1, \ldots, n-1,$$

which in turn implies (iv). We now proceed by induction. Assume α_{k+1} satisfies $A_{k+1} \leq \alpha_{k+1} \leq B_{k+1}$. Given this assumption, we need to show that $A_k \leq B_k$. Considering the

definitions (4.31) and (4.32) of A_k and B_k, this is equivalent to the following inequalities holding simultaneously:

(i) $\beta_{k+1} \le \beta_k$,

(ii) $\displaystyle\sum_{m=k}^{n}\beta_m - \sum_{m=k+1}^{n-1}\alpha_m - \mu_n \le \beta_k$,

(iii) $\displaystyle\beta_{k+1} \le \sum_{m=l}^{n-1}\mu_m - \sum_{m=l+1}^{k}\beta_m - \sum_{m=k+1}^{n-1}\alpha_m \quad \forall l = 1,\dots,k$,

(iv) $\displaystyle\sum_{m=k}^{n}\beta_m - \sum_{m=k+1}^{n-1}\alpha_m - \mu_n \le \sum_{m=l}^{n-1}\mu_m - \sum_{m=l+1}^{k}\beta_m - \sum_{m=k+1}^{n-1}\alpha_m \quad \forall l = 1,\dots,k.$

Again, the fact that $\{\beta_m\}_{m=1}^n$ is nonincreasing implies (i). Next, $\alpha_{k+1} \ge A_{k+1}$ gives:

$$\alpha_{k+1} \ge \sum_{m=k+1}^{n}\beta_m - \sum_{m=k+2}^{n-1}\alpha_m - \mu_n,$$

which is a rearrangement of (ii). Similarly, $\alpha_{k+1} \le B_{k+1}$ gives:

$$\alpha_{k+1} \le \sum_{m=l}^{n-1}\mu_m - \sum_{m=l+1}^{k+1}\beta_m - \sum_{m=k+2}^{n-1}\alpha_m \qquad \forall l = 1,\dots,k+1,$$

which is a rearrangement of (iii). Note that we don't use the fact that (iii) holds when $l = k+1$. Finally, (iv) follows from the facts that $\{\beta_m\}_{m=1}^n$ is nonincreasing and $\{\beta_m\}_{m=1}^n \ge \{\mu_m\}_{m=1}^n$, since they imply:

$$\beta_k + \sum_{m=l+1}^{n}\beta_m \le \sum_{m=l}^{n}\beta_m \le \sum_{m=l}^{n}\mu_m \qquad \forall l = 1,\dots,k,$$

which is a rearrangement of (iv). □

By starting with a sequence $\{\lambda_{N;m}\}_{m=1}^N = \{\lambda_m\}_{m=1}^M$ that majorizes given $\{\mu_m\}_{m=1}^N$, repeatedly applying Theorem 17 to construct $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$ from $\{\lambda_{n;m}\}_{m=1}^n$ results in a sequence of inner eigensteps. Conversely, any sequence of inner eigensteps $\{\{\lambda_{m;n}\}_{m=1}^n\}_{n=1}^N$ can be constructed by repeatedly applying Theorem 17. This fact is summarized in the following corollary:

Corollary 18. *Let $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$ be nonnegative and nonincreasing where $\{\lambda_n\}_{n=1}^N \succeq \{\mu_n\}_{n=1}^N$. Every sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=1}^N$ can be constructed by the following algorithm: Let $\lambda_{N;m} = \lambda_m$ for all $m = 1, \dots, N-1$; for any $n = N, \dots, 2$ construct $\{\lambda_{n-1;m}\}_{m=1}^{n-1}$ from $\{\lambda_{n;m}\}_{m=1}^n$ by picking $\lambda_{n-1;k} \in [A_{n-1;k}, B_{n-1;k}]$ for all $k = n-1, \dots, 1$, where $A_{n-1;k}$ and $B_{n-1;k}$ are given by* (4.31) *and* (4.32)*, respectively. Moreover, any sequence constructed by this algorithm is indeed a corresponding sequence of inner eigensteps.*

We now redo Example 13 to illustrate that Corollary 18 indeed gives a more systematic way of parametrizing eigensteps:

Example 19. We wish to parametrize the eigensteps corresponding to UNTFs of 5 vectors in $\mathbb{R}^3$. In the end, we will get the same parameterization of eigensteps as in Example 13:

n	1	2	3	4	5
$\lambda_{n;5}$					0
$\lambda_{n;4}$				0	0
$\lambda_{n;3}$			x	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;2}$		y	$\frac{4}{3}-x$	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	1	$2-y$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

(4.41)

where $0 \le x \le \frac{2}{3}$, $\max\{\frac{1}{3}, x\} \le y \le \min\{\frac{2}{3}+x, \frac{4}{3}-x\}$. In what follows, we rederive the above table one column at a time, in order from right to left, and filling in each column from top to bottom. First, the desired spectrum of the final Gram matrix gives us that $\lambda_{5;5} = \lambda_{5;4} = 0$ and $\lambda_{5;3} = \lambda_{5;2} = \lambda_{5;1} = \frac{5}{3}$. Next, we wish to find all $\{\lambda_{4;m}\}_{m=1}^4$ such that $\{\lambda_{4;m}\}_{m=1}^4 \sqsubseteq \{\lambda_{5;m}\}_{m=1}^5$ and $\{\lambda_{4;m}\}_{m=1}^4 \succeq \{\mu_m\}_{m=1}^4$. To this end, taking $n = 5$ and $k = 4$, Theorem 17 gives:

$$\max\{\lambda_{5;5}, \lambda_{5;4} + \lambda_{5;5} - \mu_5\} \le \lambda_{4;4} \le \min\left\{\lambda_{5;4}, \min_{l=1,\dots,4}\left\{\sum_{m=l}^{4}\mu_m - \sum_{m=l+1}^{4}\lambda_{5;m}\right\}\right\},$$

$$0 = \max\{0, -1\} \le \lambda_{4;4} \le \min\{0, \tfrac{2}{3}, \tfrac{4}{3}, 2, 1\} = 0,$$

and so $\lambda_{4;4} = 0$. For each $k = 3, 2, 1$, the same approach gives $\lambda_{4;3} = \frac{2}{3}$, $\lambda_{4;2} = \frac{5}{3}$, and $\lambda_{4;1} = \frac{5}{3}$. For the next column, we take $n = 4$. Starting with $k = 3$, we have:

$$\max\{\lambda_{4;4}, \lambda_{4;3} + \lambda_{4;4} - \mu_4\} \leq \lambda_{3;3} \leq \min\left\{\lambda_{4;3}, \min_{l=1,\dots,3}\left\{\sum_{m=l}^{3} \mu_m - \sum_{m=l+1}^{3} \lambda_{4;m}\right\}\right\},$$

$$0 = \max\{0, -\tfrac{1}{3}\} \leq \lambda_{3;3} \leq \min\{\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{4}{3}, 1\} = \tfrac{2}{3}.$$

Notice that the lower and upper bounds on $\lambda_{3;3}$ are not equal. Since $\lambda_{3;3}$ is our first free variable, we parametrize it: $\lambda_{3;3} = x$ for some $x \in [0, \frac{2}{3}]$. Next, $k = 2$ gives:

$$\tfrac{4}{3} - x = \max\{\tfrac{2}{3}, \tfrac{4}{3} - x\} \leq \lambda_{3;2} \leq \min\{\tfrac{5}{3}, \tfrac{4}{3} - x, 2 - x\} = \tfrac{4}{3} - x,$$

and so $\lambda_{3;2} = \frac{4}{3} - x$. Similarly, $\lambda_{3;1} = \frac{5}{3}$. Next, we take $n = 3$ and $k = 2$:

$$\max\{x, \tfrac{1}{3}\} \leq \lambda_{2;2} \leq \min\{\tfrac{4}{3} - x, \tfrac{2}{3} + x, 1\}.$$

Note that $\lambda_{2;2}$ is a free variable; we parametrize it as $\lambda_{2;2} = y$ such that

$$y \in [\tfrac{1}{3}, \tfrac{2}{3} + x] \text{ if } x \in [0, \tfrac{1}{3}], \qquad y \in [x, \tfrac{4}{3} - x] \text{ if } x \in [\tfrac{1}{3}, \tfrac{2}{3}].$$

Finally, $\lambda_{2;1} = 2 - y$ and $\lambda_{1;1} = 1$.

It turns out that when all $\{\mu_m\}_{m=1}^n$ are of equal lengths, the upper bound (4.32) of Theorem 17 can be simplified. In this case, the majorization requirement that $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \succeq \{\mu_m\}_{m=1}^{n-1}$ comes for free. This is a result of the fact that any nonnegative nonincreasing sequence $\{x_m\}_{m=1}^n$ of sum $s = \sum_{m=1}^n x_m$ majorizes the uniform sequence $\{u_m\}_{m=1}^n$ where $u_m = \frac{s}{n}$ for all $m = 1 \dots n$. To see this, first note for $k = 1, \dots, n$, since $\{x_m\}_{m=1}^n$ is in decreasing order, the sum of the first k elements of this sequence will always be greater than or equal to the sum of *any* k elements chosen from $\{x_m\}_{m=1}^n$. Averaging over all $\binom{n}{k}$ k-combinations, we have:

$$\sum_{m=1}^{k} x_m \geq \frac{1}{\binom{n}{k}} \sum_{m_1=1}^{n} \sum_{\substack{m_2=1 \\ m_2 \neq m_1}}^{n} \sum_{\substack{m_3=1 \\ m_3 \neq m_1, m_2}}^{n} \cdots \sum_{\substack{m_k=1 \\ m_k \neq m_1 \dots m_{k-1}}}^{n} (x_{m_1} + x_{m_2} \cdots + x_{m_k}). \tag{4.42}$$

Taking into account that each x_m is repeated a total of $\binom{n-1}{k-1}$ times, the summations in (4.42) can be collapsed into one and further simplified to give the majorization condition,

$$\sum_{m=1}^{k} x_m \geq \frac{\binom{n-1}{k-1}}{\binom{n}{k}} \sum_{m=1}^{n} x_m = \frac{(n-1)!}{(k-1)!(n-k)!} \frac{k!(n-k)!}{n!} \sum_{m=1}^{n} x_m = \frac{k}{n} \sum_{m=1}^{n} x_m = \frac{ks}{n} = \sum_{m=1}^{k} u_m,$$

which holds for $k = 1, \ldots, n$. It is straightforward to show equality in the case that $k = n$. Thus, the uniform sequence $\{u_m\}_{m=1}^n$ is indeed majorized by $\{x_n\}_{m=1}^n$. Returning to the upper bound (4.32) of Theorem 17 in the case when $\{\mu_m\}_{m=1}^n$ are all of equal lengths, we will now show that the minimum always occurs when $l = 1$. Hence, the new upper bound for $\lambda_{n-1;k}$ can be found by taking the minimum of just two quantities rather than $k + 1$ quantities.

Theorem 20. *Let $\{\lambda_{n;m}\}_{m=1}^n$ and $\{\mu_m\}_{m=1}^n$ be nonnegative nonincreasing sequences. Suppose $\sum_{m=1}^n \lambda_{n;m} = n\mu$. Then $\sum_{m=1}^{n-1} \lambda_{n-1;m} = (n-1)\mu$ and $\{\lambda_{n-1;m}\}_{m=1}^{n-1} \sqsubseteq \{\lambda_{n;m}\}_{m=1}^n$ if and only if $\lambda_{n-1;k} \in [A_{n-1;k}, B_{n-1;k}]$ for every $k = 1, \ldots, n-1$, where*

$$A_{n-1;k} := \max\Big\{\lambda_{n;k+1}, \lambda_{n;k} - \nu_{n;k}\Big\}, \tag{4.43}$$

$$B_{n-1;k} := \min\Big\{\lambda_{n;k}, \lambda_{n;1} - \nu_{n;k}\Big\}. \tag{4.44}$$

where

$$\nu_{n;k} = \mu - \sum_{m=k+1}^{n} \lambda_{n;m} + \sum_{m=k+1}^{n-1} \lambda_{n-1;m}. \tag{4.45}$$

Here, we again use the convention that sums over empty sets of indices are zero. Moreover, suppose $\lambda_{n-1;n-1}, \ldots, \lambda_{n-1;k+1}$ are consecutively chosen to satisfy these bounds. Then $A_{n-1;k} \leq B_{n-1;k}$, and so $\lambda_{n-1;k}$ can also be chosen from such an interval.

Proof. The proof here is similar to that of Theorem 17. For the sake of notational simplicity, we again let $\{\alpha_m\}_{m=1}^{n-1} := \{\lambda_{n-1;m}\}_{m=1}^{n-1}$, $\{\beta_m\}_{m=1}^n := \{\lambda_{n;m}\}_{m=1}^n$, $\nu_{n;k} = \nu_k$, $A_k := A_{n-1;k}$, and $B_k := B_{n-1;k}$.

The "⇒" direction of the proof follows immediately from Theorem 17. For the "⇐" direction, suppose $A_k \leq \alpha_k \leq B_k$ for every $k = 1, \ldots, n-1$. Then the first entries in (4.43)

and (4.44) give $\beta_{k+1} \leq \alpha_k \leq \beta_k$ for every $k = 1, \ldots, n-1$, that is, $\{\alpha_m\}_{m=1}^{n-1} \sqsubseteq \{\beta_m\}_{m=1}^{n}$, as claimed. It remains to be shown that $\sum_{m=1}^{n-1} \alpha_m = (n-1)\mu$. For $k = 1, \beta_1 - \nu_1 \leq \alpha_1 \leq \beta_1 - \nu_1$ and so

$$\alpha_1 = \beta_1 - \nu_1 = \beta_1 - \left(\mu - \sum_{m=2}^{n} \beta_m + \sum_{m=2}^{n-1} \alpha_m\right) \tag{4.46}$$

Rearranging (4.46) and using the fact that $\sum_{m=1}^{n} \beta_m = n\mu$ gives

$$\sum_{m=1}^{n-1} \alpha_m = \sum_{m=1}^{n} \beta_m - \mu = n\mu - \mu = (n-1)\mu$$

as claimed. For the final claim, we first show that the claim holds for $k = n-1$, namely that $A_{n-1} \leq B_{n-1}$. Explicitly, we need to show that

$$\max\{\beta_n, \beta_{n-1} + \beta_n - \mu\} \leq \min\{\beta_{n-1}, \beta_1 + \beta_n - \mu\}. \tag{4.47}$$

Note that (4.47) is equivalent to the following inequalities holding simultaneously:

(i) $\beta_n \leq \beta_{n-1}$,

(ii) $\beta_{n-1} + \beta_n - \mu \leq \beta_{n-1}$,

(iii) $\beta_n \leq \beta_1 + \beta_n - \mu$,

(iv) $\beta_{n-1} + \beta_n - \mu \leq \beta_1 + \beta_n - \mu$.

As in the proof of Theorem 17, (i) follows immediately from the fact that $\{\beta_m\}_{m=1}^{n}$ is nonincreasing. For (ii) and (iii), we use the fact that the average always falls between the maximum and minimum values of $\{\beta_m\}_{m=1}^{n}$:

$$\beta_n = \min_{m=1,\ldots n} \beta_m \leq \frac{1}{n} \sum_{m=1}^{n} \beta_m = \mu = \frac{1}{n} \sum_{m=1}^{n} \beta_m \leq \max_{m=1,\ldots n} \beta_m = \beta_1. \tag{4.48}$$

Items (ii) and (iii) follow immediately from (4.48).

We now proceed by induction. Assume α_{k+1} satisfies $A_{k+1} \leq \alpha_{k+1} \leq B_{k+1}$. Given this assumption, we need to show that $A_k \leq B_k$. Considering the definitions (4.43) and (4.44) of A_k and B_k, this is equivalent to the following inequalities holding simultaneously:

(i) $\beta_{k+1} \leq \beta_k$,

(ii) $\beta_k - \nu_k \leq \beta_k$,

(iii) $\beta_{k+1} \leq \beta_1 - \nu_k$,

(iv) $\beta_k - \nu_k \leq \beta_1 - \nu_k$

Again, the fact that $\{\beta_m\}_{m=1}^n$ is nonincreasing implies (i). Next, substituting the definition of ν_{k+1} from (4.45) gives

$$\begin{aligned}\nu_{k+1} - \beta_{k+1} + \alpha_{k+1} &= \mu - \sum_{m=k+2}^{n} \beta_m + \sum_{m=k+2}^{n-1} \alpha_m - \beta_{k+1} + \alpha_{k+1} \\ &= \mu - \sum_{k+1}^{n} \beta_m + \sum_{m=k+1}^{n-1} \alpha_m \\ &= \nu_k. \end{aligned} \tag{4.49}$$

Next, note that our inductive hypothesis gives $\alpha_{k+1} \geq A_{k+1} \geq \beta_{k+1} - \nu_{k+1}$. Combining this fact with (4.49) then gives $\nu_k \geq 0$, which then implies (ii). Next, substituting (4.49) into $\alpha_{k+1} \leq B_{k+1} \leq \beta_1 - \nu_{k+1}$ gives

$$0 \leq \beta_1 - \alpha_{k+1} - \nu_{k+1} = \beta_1 - \alpha_{k+1} - (\nu_k + \beta_{k+1} - \alpha_{k+1}) = \beta_1 - \nu_k - \beta_{k+1},$$

which is a rearrangement of (iii). Finally, (iv) follows from the fact that $\{\beta_m\}_{m=1}^n$ is nonincreasing. □

We now give an example to show the necessity of the minimum over all l in (4.32).

Example 21. Let $N = 4$, $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{11, 8, 8, 1\}$ and $\{\mu_1, \mu_2, \mu_3, \mu_4\} = \{10, 6, 6, 6\}$. In this example, we show that when $\{\mu_m\}_{m=1}^4$ are not of equal lengths, the minimum does not always occur when $l = 1$.

Our goal is to construct a sequence of inner eigensteps $\{\{\lambda_{n;m}\}_{m=1}^n\}_{n=1}^4$ which satisfy Definition 12. We will show that every sequence of eigensteps is given by the following

table:

$$\begin{array}{c|cccc} n & 1 & 2 & 3 & 4 \\ \hline \lambda_{n;4} & & & & 1 \\ \lambda_{n;3} & & & x & 8 \\ \lambda_{n;2} & & y & 8 & 8 \\ \lambda_{n;1} & 10 & 16-y & 14-x & 11 \end{array} \tag{4.50}$$

where $3 \leq x \leq 4$ and $2 + x \leq y \leq 6$.

We rederive this table one column at a time working backwards from right to left. To begin, the desired spectrum of the final Gram matrix completes the last column of (4.50); that is, $\lambda_{4;1} = 11$, $\lambda_{4;2} = 8$, $\lambda_{4;3} = 8$, and $\lambda_{4;4} = 1$. Next, we complete the third column of (4.50) and find all $\{\lambda_{3;m}\}_{m=1}^{3}$ such that $\{\lambda_{3;m}\}_{m=1}^{3} \sqsubseteq \{\lambda_{4;m}\}_{m=1}^{4}$ and such that $\{\lambda_{3;m}\}_{m=1}^{3} \succeq \{\mu_m\}_{m=1}^{3}$. Taking $n = 4$ and $k = 3$, Theorem 17 gives

$$\max\{\lambda_{4;4}, \lambda_{4;3} + \lambda_{4;4} - \mu_4\} \leq \lambda_{3;3} \leq \min\left\{\lambda_{4;3}, \min_{l=1,\ldots,3}\left\{\sum_{m=l}^{3}\mu_m - \sum_{m=l+1}^{3}\lambda_{4;m}\right\}\right\},$$

$$3 = \max\{1, 3\} \leq \lambda_{3;3} \leq \min\{8, 6, 4, 6\} = 4.$$

Notice that the lower and upper bounds on $\lambda_{3;3}$ are not equal. Since $\lambda_{3;3}$ is our first free variable, we parametrize it: $\lambda_{3;3} = x$ for some $x \in [3, 4]$. Also, notice that the minimum occurs at $l = 2$ which demonstrates that when $\{\mu_m\}_{m=1}^{4}$ are not of equal lengths, it is necessary to consider the minimum over all l in order for $\{\lambda_{3;m}\}_{m=1}^{3} \succeq \{\mu_m\}_{m=1}^{3}$. Indeed, if we had followed the approach of Theorem 20, the upper bound on x would occur at $l = 1$ implying $x \in [3, 6]$. However, picking $x = 6$, leads to $\lambda_{3;1} = 8$, contradicting the fact that $10 = \lambda_{1;1} \leq \lambda_{3;1}$. Finally, to complete the third column, we are left with the cases when $k = 2$ and $k = 1$. Following the approach of Theorem 17 gives $\lambda_{3;2} = 8$ and $\lambda_{3;1} = 14 - x$.

For the second column, we take $n = 3$. Starting with $k = 2$, we have

$$\max\{\lambda_{3;3}, \lambda_{3;2} + \lambda_{3;3} - \mu_3\} \leq \lambda_{2;2} \leq \min\left\{\lambda_{3;2}, \min_{l=1,\dots,2}\left\{\sum_{m=l}^{2}\mu_m - \sum_{m=l+1}^{2}\lambda_{3;m}\right\}\right\},$$

$$2 + x = \max\{x, 2 + x\} \leq \lambda_{2;2} \leq \min\{8, 8, 6\} = 6.$$

Note that $\lambda_{2;2}$ is a free variable; we parametrize it as $\lambda_{2;2} = y$ such that $2 + x \leq y \leq 6$. Finally, $\lambda_{2;1} = 16 - y$ and $\lambda_{1;1} = 10$.

In conclusion, we now give a complete constructive solution to Problem 9, that is, the problem of constructing every frame of a given spectrum and set of lengths. Recall from the beginning of this chapter that it suffices to prove Theorem 10:

Proof of Theorem 10: We first show why such an F exists if and only if we have $\{\lambda_m\}_{m=1}^{M} \cup \{0\}_{m=M+1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$. Though this may be quickly proven using the Schur-Horn Theorem—see the discussion at the beginning of Section 4.2—it also follows from the theory of this chapter. In particular, if such an F exists, then Theorem 2 implies that there exists a sequence of outer eigensteps corresponding to $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$; by Theorem 14, this implies that there exists a sequence of inner eigensteps corresponding to $\{\lambda_m\}_{m=1}^{M} \cup \{0\}_{m=M+1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$; by the discussion at the beginning of Section 4.3, we necessarily have $\{\lambda_m\}_{m=1}^{M} \cup \{0\}_{m=M+1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$. Conversely, if $\{\lambda_m\}_{m=1}^{M} \cup \{0\}_{m=M+1}^{N} \succeq \{\mu_n\}_{n=1}^{N}$, then Top Kill (Theorem 16) constructs a corresponding sequence of inner eigensteps, and so Theorem 14 implies there exists a sequence of outer eigensteps corresponding to $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$, at which point Theorem 2 implies that such an F exists.

For the remaining conclusions, note that in light of Theorem 2, it suffices to show that every valid sequence of outer eigensteps (Definition 1) satisfies the bounds of Step A of Theorem 10, and conversely, that every sequence constructed by Step A is a valid sequence of outer eigensteps. Both of these facts follow from the same two results. The first is Theorem 14, which establishes a correspondence between every valid sequence of outer eigensteps for $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$ with a valid sequence of inner eigensteps

for $\{\lambda_m\}_{m=1}^{M} \cup \{0\}_{m=M+1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$ and vice versa, the two being zero-padded versions of each other. The second relevant result is Corollary 18 which characterizes all such inner eigensteps in terms of the bounds (4.31) and (4.32) of Theorem 17. In short, the algorithm of Step A is the outer eigenstep version of the application of Corollary 18 to $\{\lambda_m\}_{m=1}^{M} \cup \{0\}_{m=M+1}^{N}$; one may easily verify that all discrepancies between the statement of Theorem 10 and Corollary 18 are the result of the zero-padding that occurs in the transition from inner to outer eigensteps.

V. Completing finite frames

In this chapter, we generalize the theory of Chapters 3 and 4 and address the problem of completing finite frames for a given nontrivial initial spectrum. We wish to complete the frame $F_N = \{f_n\}_{n=1}^{N}$ by adding P additional measurements in order to construct $F = \{f_n\}_{n=1}^{N+P}$ whose frame operator has spectrum $\{\lambda_m\}_{m=1}^{M}$. Our goal is to characterize all such frames whose final spectrum is (α, μ)-*constructible*:

Definition 22. Given nonnegative nonincreasing sequences $\alpha = \{\alpha_m\}_{m=1}^{M}$ and $\mu = \{\mu_{N+p}\}_{p=1}^{P}$, a corresponding nonnegative nonincreasing sequence $\{\lambda_m\}_{m=1}^{M}$ is (α, μ)-*constructible* if given any N vectors $F_N = \{f_n\}_{n=1}^{N}$ in $\mathbb{C}^M$ whose frame operator $F_N F_N^* = \sum_{n=1}^{N} f_n f_n^*$ has spectrum $\{\alpha_m\}_{m=1}^{M}$, it is possible to find vectors $\{f_{N+p}\}_{p=1}^{P}$ such that $\|f_{N+p}\|^2 = \mu_{N+p}$ for all $p = 1, \dots P$, and such that the new frame operator $F = \{f_n\}_{n=1}^{N+P}$ has spectrum $\{\lambda_m\}_{m=1}^{M}$.

In particular, in this chapter we solve the following problem:

Problem 23. *Given nonnegative nonincreasing sequences $\alpha = \{\alpha_m\}_{m=1}^{M}$ and $\mu = \{\mu_{N+p}\}_{p=1}^{P}$, characterize all (α, μ)-constructible sequences $\{\lambda_m\}_{m=1}^{M}$.*

Specifically, the major results of this chapter are Theorem 39 which characterizes which spectra are (α, μ)-constructible and Theorem 40 which provides the explicit Chop Kill algorithm for constructing a sequence of continued outer eigensteps.

To solve Problem 23, we build on the theory of Chapters 3 and 4 to characterize all frames whose final spectrum is (α, μ)-constructible. Taken together, the results of these two previous chapters allow us to explicitly parametrize the set of all frames whose frame operator FF^* has a given spectrum and whose elements have a given set of lengths in the case where no initial spectrum is given, i.e., where $\alpha_m = 0$ for all m. In particular, we saw that any F for which FF^* has $\{\lambda_m\}_{m=1}^{M}$ as its spectrum and for which $\|f_n\|^2 = \mu_n$ for all n generates a sequence of eigensteps. The main result of Chapter 3, Theorem 2, proves

that the converse of this statement is also true and characterizes and proves the existence of sequences of vectors that generate a given sequence of eigensteps. In light of these facts, solving Problem 23 can be reduced to finding a valid sequence of *continued outer eigensteps* which have the additional property that $\lambda_{N;m} = \alpha_m$:

Definition 24. Given nonnegative nonincreasing sequences $\{\alpha_m\}_{m=1}^M$, $\{\lambda_m\}_{m=1}^M$ and $\{\mu_{N+p}\}_{p=1}^P$, a *sequence of continued outer eigensteps* is a doubly-indexed sequence of sequences $\{\{\lambda_{N+p;m}\}_{m=1}^M\}_{p=0}^P$ for which:

(i) The initial sequence is prescribed: $\lambda_{N;m} = \alpha_m, \forall m = 1, \dots, M$.

(ii) The final sequence is $\{\lambda_m\}_{m=1}^M$: $\lambda_{N+P;m} = \lambda_m, \forall m = 1, \dots, M$.

(iii) The sequences interlace: $\{\lambda_{N+p-1;m}\}_{m=1}^M \sqsubseteq \{\lambda_{N+p;m}\}_{m=1}^M, \forall p = 1, \dots, P$.

(iv) The trace condition is satisfied: $\sum_{m=1}^M \lambda_{N+p;m} = \sum_{m=1}^M \alpha_m + \sum_{p'=1}^p \mu_{N+p'}, \forall p = 1, \dots, P$.

The differences between Definition 24 and Definition 1 of outer eigensteps in Chapter 3 are conditions (i) and (iv). Conditions (i) and (iv) are both results of the fact that we are adding measurements to the existing spectrum $\alpha = \{\alpha_m\}_{m=1}^M$. In this chapter, we are not concerned with constructing eigensteps for α as this problem has been completely resolved in Chapter 4 which provides an explicit parametrization of the set of all valid eigensteps for α. Instead, we focus our attention on constructing the continued outer eigensteps for $\{\lambda_m\}_{m=1}^M$. To this end, our goal is to complete the following table of continued outer eigensteps for given initial and final spectra:

$N+p$	0	1	2	$\dots$	P
$\lambda_{N+p;M}$	α_3	?	?	$\dots$	λ_M
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$\lambda_{N+p;1}$	α_1	?	?	$\dots$	λ_1

Note that once we have a sequence of continued outer eigensteps, we can then apply the algorithm of Theorem 7 in order to complete the frame F_N by explicitly constructing the added frame vectors, $\{f_{N+p}\}_{p=1}^{P}$.

In order to solve Problem 23 we first consider a class of sequences $\{\lambda_m\}_{m=1}^{M}$ which is larger (and easier to understand) than the (α, μ)-constructible class that we are interested in. If $\alpha = \{\alpha_m\}_{m=1}^{M}$ is the initial sequence of eigenvalues, we know from the interlacing condition of Definition 24 that $\alpha_m = \lambda_{N;m} \leq \lambda_{N+p;m}$ for $p = 1, \ldots, P$. Since α is nonnegative, $\{\lambda_{N+p;m}\}_{m=1}^{M}$ is nonnegative as well. Sequences which have these properties are what we call *α-admissible*:

Definition 25. Given a nonnegative nonincreasing sequence $\alpha = \{\alpha_m\}_{m=1}^{M}$, a corresponding nonincreasing sequence $\{\lambda_m\}_{m=1}^{M}$ is *α-admissible* if $\alpha_m \leq \lambda_m$, for all $m = 1, \ldots, M$. The set of all α-admissible sequences is denoted by $\mathrm{adm}(\alpha)$.

Every (α, μ)-constructible sequence is an α-admissible sequence, but the converse is not necessarily true. Due to the complexity of Problem 23 we first understand what it means for a sequence to be α-admissible. Once we have restricted our search to only those sequences which are α-admissible, we then tackle the problem of determining what it means for one of these sequences to be (α, μ)-constructible. To begin, in the next section we consider how only a single eigenvalue is related to a given initial sequence of eigenvalues α. In Section 5.2 we then turn our attention to solving Problem 23. There we make use of the theory in Section 5.1 to create a one-to-one correspondence between an α-admissible sequence and a new matrix which represents the relationship between the initial sequence α and final sequence $\{\lambda_m\}_{m=1}^{M}$. We also characterize sequences $\{\lambda_m\}_{m=1}^{M}$ which are (α, μ)-constructible and generalize the Top Kill algorithm of Chapter 4 for generating a valid sequence of continued outer eigensteps.

5.1 Preliminaries

In this section we explore the connection between a single eigenvalue λ and a given initial sequence of eigenvalues $\alpha = \{\alpha_m\}_{m=1}^{M}$. The following two functions will help us relate λ to α:

Definition 26. Let $\rho \in \mathbb{R}^{M+1}$. Define the function $\mathrm{s} : \mathbb{R}^{M+1} \to \mathbb{R}$

$$\mathrm{s}(\rho) := \sum_{n=1}^{M+1} \rho_n. \tag{5.1}$$

Definition 27. Let $\lambda \in [0, \infty)$. Given a nonnegative nonincreasing sequence $\alpha = \{\alpha_m\}_{m=1}^{M}$, define $\mathrm{p}_\alpha : [0, \infty) \to \mathbb{R}^{M+1}$

$$[\mathrm{p}_\alpha(\lambda)]_n := \ell([0, \lambda] \cap (\alpha_n, \alpha_{n-1}]), \tag{5.2}$$

where ℓ is the length (Lebesgue measure) of an interval and we adopt the convention that $\alpha_0 := \infty$ and $\alpha_{M+1} := 0$.

The function s simply computes the sum of the elements of any vector whereas p_α can be thought of as taking any nonnegative real number, λ, and breaking it up into intervals according to α. In other words, p_α partitions λ according to α leading to the following definition.

Definition 28. Define the set of all *α-partitions* as follows:

$$\mathrm{Part}(\alpha) := \left\{ \rho \in \mathbb{R}^{M+1} : \begin{array}{l} 0 \le \rho_n \le \alpha_{n-1} - \alpha_n, \forall n; \\ \text{if } \rho_n > 0, \text{ then } \rho_{n'} = \alpha_{n'-1} - \alpha_{n'}, \forall n' > n \end{array} \right\}. \tag{5.3}$$

In a moment, we will show that $\mathrm{Part}(\alpha)$ is the range of p_α. The next result states that any vector in $\mathrm{Part}(\alpha)$ must be of a particular form.

Lemma 29. *Given a nonnegative nonincreasing sequence $\alpha = \{\alpha_m\}_{m=1}^{M}$ and $\rho \in \mathrm{Part}(\alpha)$, there exists an index $n_0 = 1 \ldots M+1$ such that:*

$$\rho_n = \begin{cases} 0, & \text{if } n < n_0, \\ \mathrm{s}(\rho) - \alpha_n, & \text{if } n = n_0, \\ \alpha_{n-1} - \alpha_n, & \text{if } n > n_0. \end{cases} \tag{5.4}$$

Specifically, for nonzero $\rho \in \mathrm{Part}(\alpha)$, n_0 is the unique index such that $\alpha_{n_0} < \mathrm{s}(\rho) \leq \alpha_{n_0-1}$.

Proof. Let $\rho = \{\rho_n\}_{n=1}^{M+1} \in \mathrm{Part}(\alpha)$. We claim that in the case where $\rho_n = 0$ for all $n = 1, \ldots, M+1$, we can choose $n_0 = M+1$. Indeed, we immediately have $\rho_n = 0$ for $n < M+1$. Moreover, for $n = M+1$, $\rho_n = \mathrm{s}(\rho) - \alpha_n = \mathrm{s}(0) - \alpha_{M+1} = 0$ since by definition, $\alpha_{M+1} := 0$. Hence $\rho = 0$ is of form (5.4).

Now consider the case where $\rho \neq 0$. Pick n_0 to be the minimal index such that $\rho_{n_0} > 0$. Since n_0 is the *minimal* such index, $\rho_n = 0$ for all $n < n_0$. Moreover, by definition of $\mathrm{Part}(\alpha)$, $\rho_n = \alpha_{n-1} - \alpha_n$ for all $n > n_0$. In light of these facts,

$$\begin{aligned} \rho_{n_0} &= \sum_{n=1}^{M+1} \rho_n - \sum_{n=1}^{n_0-1} \rho_n - \sum_{n=n_0+1}^{M+1} \rho_n \\ &= \mathrm{s}(\rho) - 0 - \sum_{n=n_0+1}^{M+1} (\alpha_{n-1} - \alpha_n) \\ &= \mathrm{s}(\rho) - (\alpha_{n_0} - \alpha_{M+1}) \\ &= \mathrm{s}(\rho) - \alpha_{n_0}, \end{aligned}$$

where the final equality follows from the fact that $\alpha_{M+1} := 0$. Thus, ρ is of form (5.4).

Next we show that the index n_0 is unique provided $\rho \neq 0$. Take any n_0 such that (5.4) holds. First note that since $\mathrm{s}(\rho) - \alpha_{n_0} = \rho_{n_0} > 0$, this gives that $\alpha_{n_0} < \mathrm{s}(\rho)$. Moreover, since $\rho_n = 0$ for all $n < n_0$ and $\rho_n \leq (\alpha_{n-1} - \alpha_n)$ for all n by (5.3), we have:

$$\mathrm{s}(\rho) = \sum_{n=1}^{M+1} \rho_n = \sum_{n=n_0}^{M+1} \rho_n \leq \sum_{n=n_0}^{M+1} (\alpha_{n-1} - \alpha_n) = \alpha_{n_0-1} - \alpha_{M+1} = \alpha_{n_0-1}.$$

Putting these two facts together yields $\alpha_{n_0} < \mathrm{s}(\rho) \le \alpha_{n_0 - 1}$. This implies that the index n_0 is unique since such an inequality can only hold for at most one n_0. $\square$

In the next section, we will see that solving Problem 23 will require a one-to-one correspondence between any eigenvalue λ_m and its α-partition $\mathrm{p}_\alpha(\lambda_m)$. The following theorem states that p_α is a bijection from the nonnegative real numbers onto their corresponding α-partitions.

Theorem 30. p_α *is a bijection from* $[0, \infty)$ *onto* $\mathrm{Part}(\alpha)$*, with*

$$\mathrm{p}_\alpha{}^{-1} = \mathrm{s} : \mathrm{Part}(\alpha) \to [0, \infty).$$

Proof. We first show that s is a left-inverse of p_α for any $\lambda \ge 0$:

$$\begin{aligned} \mathrm{s}(\mathrm{p}_\alpha(\lambda)) = \sum_{n=1}^{M+1} [\mathrm{p}_\alpha(\lambda)]_n &= \sum_{n=1}^{M+1} \ell([0, \lambda] \cap (\alpha_n, \alpha_{n-1}]) \\ &= \ell\left(\bigcup_{n=1}^{M+1} [0, \lambda] \cap (\alpha_n, \alpha_{n-1}] \right) \\ &= \ell\left([0, \lambda] \cap \left(\bigcup_{n=1}^{M+1} (\alpha_n, \alpha_{n-1}] \right) \right). \end{aligned} \tag{5.5}$$

By definition, $\alpha_0 := \infty$ and $\alpha_{M+1} := 0$, implying $\bigcup_{n=1}^{M+1} (\alpha_n, \alpha_{n-1}] = (0, \infty]$. Coupled with the fact that $\lambda \ge 0$, (5.5) becomes

$$\mathrm{s}(\mathrm{p}_\alpha(\lambda)) = \ell([0, \lambda] \cap (0, \infty]) = \ell((0, \lambda]) = \lambda.$$

Thus, s is indeed a left-inverse of p_α, in particular implying p_α is one-to-one.

To show that s is a right-inverse of p_α, let $\rho = \{\rho_n\}_{n=1}^{M+1} \in \mathrm{Part}(\alpha)$, and without loss of generality, assume $\rho \ne 0$. Then by Definition 27,

$$[\mathrm{p}_\alpha(\mathrm{s}(\rho))]_n = \ell([0, \mathrm{s}(\rho)] \cap (\alpha_n, \alpha_{n-1}]) = \begin{cases} 0, & \text{if } \mathrm{s}(\rho) \le \alpha_n, \\ \mathrm{s}(\rho) - \alpha_n, & \text{if } \alpha_n < \mathrm{s}(\rho) \le \alpha_{n-1}, \\ \alpha_{n-1} - \alpha_n, & \text{if } \alpha_{n-1} < \mathrm{s}(\rho). \end{cases} \tag{5.6}$$

Moreover, since $\rho \neq 0$, the n_0 of Lemma 29 is the unique index such that $\alpha_{n_0} < \mathrm{s}(\rho) \leq \alpha_{n_0-1}$. We use this fact to prove that $\mathrm{p}_\alpha(\mathrm{s}(\rho)) = \rho$ by relating (5.6) to (5.4). That is, we want to show that

$$\left\{\begin{array}{ll} 0 & \text{if } \mathrm{s}(\rho) \leq \alpha_n, \\ \mathrm{s}(\rho) - \alpha_n, & \text{if } \alpha_n < \mathrm{s}(\rho) \leq \alpha_{n-1}, \\ \alpha_{n-1} - \alpha_n, & \text{if } \alpha_{n-1} < \mathrm{s}(\rho). \end{array}\right\} = \left\{\begin{array}{ll} 0 & \text{if } n < n_0 \\ \mathrm{s}(\rho) - \alpha_n, & \text{if } n = n_0, \\ \alpha_{n-1} - \alpha_n, & \text{if } n > n_0. \end{array}\right\}. \tag{5.7}$$

In particular, considering (5.7) for an index n for which $\mathrm{s}(\rho) \leq \alpha_n$, (5.6) gives $[\mathrm{p}_\alpha(\mathrm{s}(\rho))]_n = 0$. Moreover, since $\alpha_{n_0} < \mathrm{s}(\rho) \leq \alpha_n$ and $\{\alpha_m\}_{m=1}^M$ is nonincreasing, $n < n_0$ and so (5.4) gives $\rho_n = 0$. Meanwhile, for n such that $\mathrm{s}(\rho) > \alpha_{n-1}$, (5.6) gives $[\mathrm{p}_\alpha(\mathrm{s}(\rho))]_n = \alpha_{n-1} - \alpha_n$ while (5.4) gives $\rho_n = \alpha_{n-1} - \alpha_n$ since the fact that $\alpha_{n-1} < \mathrm{s}(\rho) \leq \alpha_{n_0-1}$ implies $n > n_0$. In the final case where n happens to satisfy $\alpha_n < \mathrm{s}(\rho) \leq \alpha_{n-1}$, we necessarily have that $n = n_0$—the index n_0 in Lemma 29 is unique when $\rho \leq 0$—implying $[\mathrm{p}_\alpha(\mathrm{s}(\rho))]_n = \mathrm{s}(\rho) - \alpha_{n_0} = \rho_n$. Having that $\mathrm{s} : [0, \infty) \to \mathrm{Part}(\alpha)$ is both a left and right inverse of p_α, then we have p_α is a bijection from $[0, \infty)$ onto $\mathrm{Part}(\alpha)$. □

The following example demonstrates how we use the function p_α to go back and forth between a single eigenvalue λ and its α-partition.

Example 31. Let $M = 3$, $\lambda = \frac{13}{4}$ and $\{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$. The α-partition of λ, $\mathrm{p}_\alpha(\lambda)$, is the 1×4 vector where each entry in computed using Definition 27. That is,

$$[\mathrm{p}_\alpha(\lambda)]_n = \ell([0, \lambda] \cap (\alpha_n, \alpha_{n-1}]).$$

When $n = 1$, by definition, $\alpha_0 := \infty$ and so

$$[\mathrm{p}_\alpha(\lambda)]_1 = \ell([0, \lambda] \cap (\alpha_1, \alpha_0]) = \ell([0, \tfrac{13}{4}] \cap (\tfrac{7}{4}, \infty]) = \ell((\tfrac{7}{4}, \tfrac{13}{4}]) = \tfrac{3}{2}. \tag{5.8}$$

Continuing for $n = 2$ and $n = 3$,

$$\begin{aligned} [\mathrm{p}_\alpha(\lambda)]_2 &= \ell([0, \lambda] \cap (\alpha_2, \alpha_1]) = \ell([0, \tfrac{13}{4}] \cap (\tfrac{3}{4}, \tfrac{7}{4}]) = \ell((\tfrac{3}{4}, \tfrac{7}{4}]) = 1, \\ [\mathrm{p}_\alpha(\lambda)]_3 &= \ell([0, \lambda] \cap (\alpha_3, \alpha_2]) = \ell([0, \tfrac{13}{4}] \cap (\tfrac{1}{2}, \tfrac{3}{4}]) = \ell((\tfrac{1}{2}, \tfrac{3}{4}]) = \tfrac{1}{4}. \end{aligned} \tag{5.9}$$

Finally in the case that $n = 4$, we use the fact that $\alpha_4 := 0$ which gives

$$[\mathrm{p}_\alpha(\lambda)]_4 = \ell([0,\lambda] \cap (\alpha_5, \alpha_4]) = \ell([0, \tfrac{13}{4}] \cap (0, \tfrac{1}{2}]) = \ell((0, \tfrac{1}{2}]) = \tfrac{1}{2}. \tag{5.10}$$

Taking (5.8), (5.9), and (5.10) together gives that the α partition of $\lambda = \frac{13}{4}$ is

$$[\mathrm{p}_\alpha(\tfrac{13}{4})] = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{5.11}$$

It is straightforward to verify that (5.11) is also consistent with Lemma 29 when $n_0 = 1$.

In the next section, we use these results to relate the initial spectrum $\{\alpha_m\}_{m=1}^M$ to the final one, $\{\lambda_m\}_{m=1}^M$.

5.2 (α, μ)-constructible sequences and Chop Kill

In this section, we focus on Problem 23, namely the problem of finding all (α, μ)-constructible sequences. Solving this problem involves finding a valid sequence of continued outer eigensteps (Definition 24) for given nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^M$, $\alpha = \{\alpha_m\}_{m=1}^M$ and $\mu = \{\mu_{N+p}\}_{p=1}^P$. Just as in Chapter 4, we can view the problem of constructing a sequence of continued outer eigensteps as building a staircase—the difference here being that we already have an initial sequence of eigenvalues, or stairsteps, to build on.

When no initial spectrum is given, we have already seen that the highest levels are the hardest the build. This motivated the Top Kill algorithm for constructing a sequence of inner eigensteps by removing as much area from the top of the stairsteps as possible since these areas require the most foresight to build. In that case, a sequence of inner eigensteps existed if and only if $\{\lambda_n\}_{n=1}^N$ majorized $\{\mu_n\}_{n=1}^N$ where $\{\lambda_n\}_{n=1}^N$, the spectrum of F^*F, is a zero-padded version of $\{\lambda_m\}_{m=1}^M$ when $M \leq N$. In contrast, when we consider the current problem of constructing continued outer eigensteps on top of a given nontrivial initial spectrum, the areas that are the hardest to build are those that are the highest *relative* to the initial spectrum α. In this setting, determining whether a sequence of eigensteps

exists is not as straightforward. In this section, we solve Problem 23 by introducing a new algorithm, the Chop Kill algorithm, for generating a valid sequence of continued outer eigensteps when starting from a nontrivial initial spectrum α. In order to solve this problem we build on the theory of Section 5.1 in order to relate the initial spectrum α to the final spectrum $\{\lambda_m\}_{m=1}^M$. First, consider the following definitions which are vectorized version of Definitions 26 and 27 in Section 5.1.

Definition 32. Let $\Lambda \in \mathbb{R}^{M\times(M+1)}$. Define the function $\mathrm{S} : \mathbb{R}^{M\times(M+1)} \to \mathbb{R}^M$,

$$[\mathrm{S}(\Lambda)]_m := \mathrm{s}(\Lambda_{m,:}). \tag{5.12}$$

Definition 33. Given a nonnegative nonincreasing sequence $\alpha = \{\alpha_m\}_{m=1}^M$ let $\{\lambda_m\}_{m=1}^M$ be α-admissible. Define the function $\mathrm{P}_\alpha : \mathrm{adm}(\alpha) \to \mathbb{R}^{M\times(M+1)}$,

$$[\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)]_{m,n} := [\mathrm{p}_\alpha(\lambda_m)]_n = \ell([0, \lambda_m] \cap (\alpha_n, \alpha_{n-1}]), \tag{5.13}$$

where p_α is the function defined in Definition 27.

The function S operates by applying s defined in Section 5.1 to each row of the matrix Λ. That is, S sums across rows. Similarly, P_α operates by applying p_α to each element of $\{\lambda_m\}_{m=1}^M$ and then stacking the outputs as rows of some $M \times (M+1)$ matrix Λ. That is, the mth row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)$ represents how the eigenvalue λ_m is broken up into intervals according to the initial spectrum α. Because of this, we refer to $\Lambda = \mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)$ as the *spectral partition matrix* of $\{\lambda_m\}_{m=1}^M$. The following example demonstrates how to find the spectral partition matrix for a set of eigenvalues.

Example 34. Let $M = 3$. We now use Definition 33 to construct the 3×4 spectral partition matrix $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$ where $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$, and $\{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{2}, \frac{1}{2}\}$. As already mentioned, P_α operates by applying p_α to each element of $\{\lambda_m\}_{m=1}^3$ and then stacking the outputs as rows of the matrix $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$. As such, we calculate the entries of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$ one row at a time.

When $m = 1$, the entries of the first row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$ correspond to the α-partition of $\lambda_1 = \frac{13}{4}$. Using the result of Example 31 where we calculated the α-partition for $\lambda = \frac{13}{4}$, the first row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$ is given by (5.11):

$$[\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{1,:} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{5.14}$$

Next, for $m = 2$, by Definition 33, each entry of the second row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$ is given by

$$[\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{2,n} = \ell([0, \lambda_2] \cap (\alpha_n, \alpha_{n-1}]), \tag{5.15}$$

where $n = 1, \ldots, 4$. Substituting $n = 1, \ldots, 4$, into (5.15) and using the fact the $\alpha_0 := \infty$ and $\alpha_4 := 0$, gives

$$\begin{aligned} [\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{2,1} &= \ell([0, \tfrac{9}{4}] \cap (\tfrac{7}{4}, \infty]) = \ell((\tfrac{7}{4}, \tfrac{9}{4}]) = \tfrac{1}{2}, \\ [\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{2,2} &= \ell([0, \tfrac{9}{4}] \cap (\tfrac{3}{4}, \tfrac{7}{4}]) = \ell((\tfrac{3}{4}, \tfrac{7}{4}]) = 1, \\ [\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{2,3} &= \ell([0, \tfrac{9}{4}] \cap (\tfrac{1}{2}, \tfrac{3}{4}]) = \ell((\tfrac{1}{2}, \tfrac{3}{4}]) = \tfrac{1}{4}, \\ [\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{2,4} &= \ell([0, \tfrac{9}{4}] \cap (0, \tfrac{1}{2}]) = \ell((0, \tfrac{1}{2}]) = \tfrac{1}{2}. \end{aligned} \tag{5.16}$$

Collectively, the equations of (5.16) give the second row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$:

$$[\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{2,:} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{5.17}$$

Repeating the calculations for $\lambda_3 = 1$, the last row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)$ becomes

$$[\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3)]_{3,:} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{5.18}$$

Stacking (5.14), (5.17), and (5.18) together, the spectral partition matrix for $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$ is given by:

$$\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^3) = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{5.19}$$

Note that by definition, each row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)$ is an α-partition. We will show in a moment that a one-to-one correspondence exists between any α-admissible sequence (Definition 25) and the spectral partition matrix it generates, $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)$. Since we will show that we are able to go back and forth between the sequence $\{\lambda_m\}_{m=1}^M$ and its spectral partition matrix, $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)$, we use the same term α-admissible to refer to a specific class of matrices as well:

Definition 35. Given a nonnegative nonincreasing sequence $\alpha = \{\alpha_m\}_{m=1}^M$ the $M \times (M+1)$ matrix Λ is *α-admissible* if:

(i) each row of Λ is an α-partition, i.e., lies in the set $\mathrm{Part}(\alpha)$ defined in (5.3).

(ii) the values in any column of Λ appear in decreasing order, i.e., $\Lambda_{m,n} \le \Lambda_{m-1,n}$ for all $m = 2 \dots M$ and $n = 1, \dots, M+1$.

(iii) $\Lambda_{m,n} = \alpha_{n-1} - \alpha_n$ whenever $n > m$.

The set of all α-admissible matrices is denoted by $\mathrm{Adm}(\alpha)$.

We now prove that P_α is a bijection from $\mathrm{adm}(\alpha)$ onto $\mathrm{Adm}(\alpha)$.

Theorem 36. P_α *is a bijection from* $\mathrm{adm}(\alpha)$ *onto* $\mathrm{Adm}(\alpha)$ *with*

$$\mathrm{P}_\alpha{}^{-1} = \mathrm{S} : \mathrm{Adm}(\alpha) \to \mathrm{adm}(\alpha).$$

Proof. Let $\alpha = \{\alpha_m\}_{m=1}^M$ be nonnegative and nonincreasing. We first show that P_α is well-defined, namely that if $\{\lambda_m\}_{m=1}^M \in \mathrm{adm}(\alpha)$ then $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M) \in \mathrm{Adm}(\alpha)$. First note that since $\mathrm{Part}(\alpha)$ is defined to be the range of p_α, each row of $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)$ is an α-partition and so (i) of Definition 35 is satisfied. For (ii), since $\{\lambda_m\}_{m=1}^M$ is nonincreasing, $[0, \lambda_m] \subseteq [0, \lambda_{m-1}]$ for all $m = 2, \dots M$, which in turn implies that

$$[\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)]_{m,n} = \ell([0, \lambda_m] \cap (\alpha_n, \alpha_{n-1}]) \le \ell([0, \lambda_{m-1}] \cap (\alpha_n, \alpha_{n-1}]) = [\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)]_{m-1,n}$$

for all $n = 2, \ldots, M+1$. Finally for (iii), for any $n > m$, we have that $n-1 \geq m$ which gives $\lambda_m \geq \lambda_{n-1} \geq \alpha_{n-1}$, where the last inequality follows from the fact that $\{\lambda_m\}_{m=1}^M \in \mathrm{adm}(\alpha)$. In this case, (5.13) becomes

$$[\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)]_{m,n} = \ell([0, \lambda_m] \cap (\alpha_n, \alpha_{n-1}]) = \ell((\alpha_n, \alpha_{n-1}]) = \alpha_{n-1} - \alpha_n.$$

Hence, $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M) \in \mathrm{Adm}(\alpha)$ and so P_α is into $\mathrm{Adm}(\alpha)$. Next we show that S is well-defined, namely that if $\Lambda \in \mathrm{Adm}(\alpha)$ then $\mathrm{S}(\Lambda) \in \mathrm{adm}(\alpha)$. Here, for any $\Lambda \in \mathrm{Adm}(\alpha)$, using the fact that $\Lambda_{m,n} \leq \Lambda_{m-1,n}$ from Definition 35.(ii),

$$[\mathrm{S}(\Lambda)]_m = \sum_{n=1}^{M+1} \Lambda_{m,n} \leq \sum_{n=1}^{M+1} \Lambda_{m-1,n} = [\mathrm{S}(\Lambda)]_{m-1},$$

which implies $\{\mathrm{S}(\Lambda)_m\}_{m=1}^M$ is nonincreasing. Additionally, from property (iii) of Definition 35,

$$[\mathrm{S}(\Lambda)]_m = \sum_{n=1}^{M+1} \Lambda_{m,n} \geq \sum_{n=m+1}^{M+1} \Lambda_{m,n} = \sum_{n=m+1}^{M+1} (\alpha_{n-1} - \alpha_n) = \alpha_m - \alpha_{M+1} = \alpha_m.$$

Putting together the fact that $\{\mathrm{S}(\Lambda)_m\}_{m=1}^M$ is nonincreasing with the fact that $[\mathrm{S}(\Lambda)]_m \geq \alpha_m$ for all $m = 1, \ldots, M$, we have $\{\mathrm{S}(\Lambda)_m\}_{m=1}^M$ is α-admissible. Moreover, we claim that P_α and S are inverses of each other over these domains. This follows from the fact that P_α and S are simply vectorized versions of p_α and s, which were proved to be inverses of each other in Theorem 36. □

Now that we have a one-to-one correspondence between any sequence of eigenvalues $\{\lambda_m\}_{m=1}^M$, and its spectral partition matrix $\mathrm{P}_\alpha(\{\lambda_m\}_{m=1}^M)$, we simplify notation by denoting this matrix as simply Λ for the remainder of the chapter. Now we turn our attention to Problem 23, specifically characterizing all (α, μ)-constructible sequences. Later on, we will see that the lower diagonals of Λ play an important role in determining whether or not a sequence $\{\lambda_m\}_{m=1}^M$ is (α, μ)-constructible. For each pair of indices in $\mathcal{M} = \{(m, n) : m = 1, \ldots, M, n = 1, \ldots, m\}$, we now introduce a new indexing which indexes each of these

lower triangular elements according to the diagonal it belongs to as well as its location on the diagonal. In this case, we use the subscript $\Lambda_{\sigma(j,k)}$ where $j = 1, \ldots, M$ is for the diagonal and $k = 1, \ldots, M - j + 1$ is the location along that diagonal. We let

$$\mathcal{J} := \{(j, k) : j = 1, \ldots, M, k = 1, \ldots, M - j + 1\}$$

be the set of all pairs of indices in the lower triangle of Λ. Given $\Lambda_{\sigma(j,k)}$, we can go back and forth between its corresponding row and column index $(m, n) \in \mathcal{M}$ as a function of j and k,

$$(m, n) = \sigma(j, k) = (j - 1 + k, k). \tag{5.20}$$

Conversely, given $\Lambda_{m,n}$ we can determine its lower diagonal and location index $(j, k) \in \mathcal{J}$ as a function of m and n,

$$(j, k) = \tau(m, n) = (m - n + 1, n). \tag{5.21}$$

For example, if $M = 3$, indexing the 3×4 matrix Λ according to standard row and column indexing, the (m, n) entry of Λ is assigned the label

$$\begin{bmatrix} (1,1) & * & * & * \\ (2,1) & (2,2) & * & * \\ (3,1) & (3,2) & (3,3) & * \end{bmatrix} = \begin{bmatrix} \sigma(1,1) & * & * & * \\ \sigma(2,1) & \sigma(1,2) & * & * \\ \sigma(3,1) & \sigma(2,2) & \sigma(1,3) & * \end{bmatrix}. \tag{5.22}$$

It it straightforward to show that the coordinate transforms (5.20) and (5.21) are inverse bijections from $\mathcal{M}$ to $\mathcal{J}$. We now define the following function which computes the sum of any lower diagonal of Λ:

Definition 37. Let $\Lambda \in \mathbb{R}^{M\times(M+1)}$. Define the *diagonal sum* function $\mathrm{DS} : \mathbb{R}^{M\times(M+1)} \to \mathbb{R}^M$

$$\mathrm{DS}(\Lambda)_j := \sum_{k=1}^{M-j+1} \Lambda_{\sigma(j,k)} = \sum_{k=1}^{M-j+1} \Lambda_{j-1+k,k}. \tag{5.23}$$

The function DS computes sums along the diagonals of the lower triangular part of $\Lambda \in \mathbb{R}^{M\times(M+1)}$. For any Λ, we claim that $\mathrm{DS}(\Lambda)$ is a nonnegative nonincreasing sequence.

Indeed, DS(Λ) is nonnegative as a result of Property (i) of Definition 35 which requires each row of Λ to be an α-partition, meaning all entries of Λ are nonnegative. Meanwhile, the fact that DS(Λ) is nonincreasing follows from property (ii), particularly that $\Lambda_{m,n} \leq \Lambda_{m-1,n}$. Specifically, for $j = 2, \ldots, M$ we have,

$$[\mathrm{DS}(\Lambda)]_j = \sum_{k=1}^{M-j+1} \Lambda_{j-1+k,k} \leq \sum_{k=1}^{M-(j-1)} \Lambda_{(j-1)-1+k,k} \leq [\mathrm{DS}(\Lambda)]_{j-1}.$$

The following example demonstrates how to calculate the diagonal sums of the spectral partition matrix (5.19) of Example 34.

Example 38. Recall from Example 34, the spectral partition matrix for $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$ and $\{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{2}, \frac{1}{2}\}$:

$$\Lambda = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

We now use Definition 37 to calculate the lower diagonal sums of Λ. There are three lower diagonals of Λ: DS$(\Lambda)_1$ consists of three elements, DS$(\Lambda)_2$ consists of two elements, and DS$(\Lambda)_3$ consists of only one element. When $j = 1$, (5.23) gives

$$\mathrm{DS}(\Lambda)_1 = \sum_{k=1}^{3} \Lambda_{\sigma(1,k)} = \sum_{k=1}^{3} \Lambda_{k,k} = \Lambda_{1,1} + \Lambda_{2,2} + \Lambda_{3,3} = \tfrac{3}{2} + 1 + \tfrac{1}{4} = \tfrac{11}{4}.$$

Similarly, when $j = 2$, (5.23) gives

$$\mathrm{DS}(\Lambda)_2 = \sum_{k=1}^{2} \Lambda_{\sigma(2,k)} = \sum_{k=1}^{3} \Lambda_{1+k,k} = \Lambda_{2,1} + \Lambda_{3,2} = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}.$$

Finally, when $j = 3$, DS$(\Lambda)_3$ only consists of one element, and so $\mathrm{DS}(\Lambda_3) = \Lambda_{\sigma(3,1)} = \Lambda_{3,1} = 0$. Notice that $\mathrm{DS}(\Lambda) = \{\frac{13}{4}, \frac{3}{4}, 0\}$ is a nonnegative nonincreasing sequence as claimed.

As previously mentioned, the lower diagonals of Λ play an important role in determining whether or not a sequence $\{\lambda_m\}_{m=1}^{M}$ is (α, μ)-constructible. In fact, we now show that if a sequence $\{\lambda_m\}_{m=1}^{M}$ is (α, μ)-constructible $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^{M}$ necessarily majorizes $\{\mu_{N+p}\}_{p=1}^{P}$.

Theorem 39. *Given nonnegative nonincreasing sequences $\{\alpha_m\}_{m=1}^M$, $\{\mu_{N+p}\}_{p=1}^P$, and $\{\lambda_m\}_{m=1}^M$, which satisfies $\lambda_m \geq \alpha_m$ for all $m = 1, \ldots, M$, if $\{\lambda_m\}_{m=1}^M$ is (α, μ)-constructible then* $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^M \geq \{\mu_{N+p}\}_{p=1}^P$.

Proof. Let $\alpha = \{\alpha_m\}_{m=1}^M$, $\mu = \{\mu_{N+p}\}_{p=1}^P$ be nonnegative nonincreasing sequences and $\{\lambda_m\}_{m=1}^M$ be α-admissible. If $\{\lambda_m\}_{m=1}^M$ is (α, μ)-constructible, there exists a sequence of continued outer eigensteps $\{\{\lambda_{N+p;m}\}_{m=1}^M\}_{p=0}^P$ and a corresponding sequence $\{\Lambda_{N+p}\}_{p=0}^P$ of α-admissible matrices where $\Lambda_{N+p} = \mathrm{P}_\alpha(\{\lambda_{N+p;m}\}_{m=1}^M)$ is the spectral partition matrix of $\{\lambda_{N+p;m}\}_{m=1}^M$, and $\Lambda := \Lambda_{N+P}$. We claim that

$$\sum_{p'=1}^{P} \mu_{N+p'} = \sum_{j=1}^{M} \mathrm{DS}(\Lambda)_j, \tag{5.24}$$

$$\sum_{p'=1}^{p} \mu_{N+p'} \leq \sum_{j=1}^{\min\{M,p\}} \mathrm{DS}(\Lambda)_j, \tag{5.25}$$

for all $p = 1, \ldots, P$. From the trace condition (iii) of continued outer eigensteps and the convention that $\alpha_{M+1} = 0$, we have

$$\begin{aligned} \sum_{p'=1}^{p} \mu_{N+p'} &= \sum_{m=1}^{M} \lambda_{N+p;m} - \sum_{m=1}^{M} \alpha_m \\ &= \sum_{m=1}^{M} \Big(\lambda_{N+p;m} - (\alpha_m - \alpha_{M+1}) \Big) \\ &= \sum_{m=1}^{M} \Big(\lambda_{N+p;m} - \sum_{n=m+1}^{M+1} (\alpha_{n-1} - \alpha_n) \Big). \end{aligned} \tag{5.26}$$

Since $\{\lambda_{N+p;m}\}_{m=1}^M$ is α-admissible for any p, Theorem 36 gives that $\Lambda_{N+p} \in \mathrm{Adm}(\alpha)$. Property (iii) of Definition 35 then gives that $\Lambda_{N+p;m,n} = \alpha_{n-1} - \alpha_n$ whenever $n > m$. As such, (5.26) can be rewritten as

$$\sum_{p'=1}^{p} \mu_{N+p'} = \sum_{m=1}^{M} \Big(\lambda_{N+p;m} - \sum_{n=m+1}^{M+1} \Lambda_{N+p;m,n} \Big). \tag{5.27}$$

Combining this with the fact that $\lambda_{N+p;m} = \sum_{n=1}^{M+1} \Lambda_{N+p;m,n}$, (5.27) becomes

$$\sum_{p'=1}^{p} \mu_{N+p'} = \sum_{m=1}^{M} \Big(\sum_{n=1}^{M+1} \Lambda_{N+p;m,n} - \sum_{n=m+1}^{M+1} \Lambda_{N+p;m,n} \Big) = \sum_{m=1}^{M} \sum_{n=1}^{m} \Lambda_{N+p;m,n}, \tag{5.28}$$

which is a sum of all the lower triangular elements of Λ_{N+p}. By making a change of variables according to (5.21), these lower triangular elements can also be summed along diagonals. That is, (5.28) becomes

$$\sum_{p'=1}^{p}\mu_{N+p'} = \sum_{m=1}^{M}\sum_{n=1}^{m}\Lambda_{N+p;m,n} = \sum_{j=1}^{M}\sum_{k=1}^{M-j+1}\Lambda_{N+p;j-1+k,k} = \sum_{j=1}^{M}\mathrm{DS}(\Lambda_{N+p})_j. \tag{5.29}$$

Letting $p = P$, gives the majorization condition (5.24). In order to obtain the remaining condition (5.25) for $p = 1,\dots,P-1$, we first show that $\mathrm{DS}(\Lambda_{N+p})_j = 0$ whenever $j > p$. To do this, note that for any $k = 1,\dots,M-j+1$, by Definition 33,

$$\Lambda_{N+p;j-1+k,k} = [\mathrm{p}_\alpha(\lambda_{N+p;j-1+k})]_k = \ell([0,\lambda_{N+p;j-1+k}]\cap(\alpha_k,\alpha_{k-1}]). \tag{5.30}$$

Since the continued outer eigensteps $\{\{\lambda_{N+p;m}\}_{m=1}^{M}\}_{p=1}^{P}$ must interlace (Definition 24.(iii)), we have

$$\lambda_{N+p;j-1+k} \le \lambda_{N+p-1;j-2+k} \le \cdots \le \lambda_{N;j-1-p+k} = \alpha_{j-1-p+k}, \tag{5.31}$$

for $M \ge j-1-p+k \ge 1$. Continuing, since $j > p$, we have $j-1 \ge p$ and so $j-1-p+k \ge p-p+k = k$. Thus, (5.31) becomes $\lambda_{N+p;j-1+k} \le \alpha_k$, at which point (5.30) becomes zero, and so

$$\mathrm{DS}(\Lambda_{N+p})_j = \sum_{k=1}^{M-j+1}\Lambda_{N+p;j-1+k,k} = 0,$$

whenever $j > p$, as claimed. Returning to (5.29), the fact that $\mathrm{DS}(\Lambda_{N+p})_j = 0$ whenever $j > p$ gives

$$\sum_{p'=1}^{p}\mu_{N+p'} = \sum_{j=1}^{M}\mathrm{DS}(\Lambda_{N+p})_j = \sum_{j=1}^{\min\{p,M\}}\mathrm{DS}(\Lambda_{N+p})_j. \tag{5.32}$$

In order to obtain the final majorization condition (5.25), we also claim that $\mathrm{DS}(\Lambda_{N+p})_j \le \mathrm{DS}(\Lambda_{N+p+1})_j$ for $p = 0,\dots,P-1$ and $j = 1,\dots,M$. To see this, we use the fact that $\{\lambda_{N+p;m}\}_{m=1}^{M} \sqsubseteq \{\lambda_{N+p+1;m}\}_{m=1}^{M}$, specifically that $\lambda_{N+p;m} \le \lambda_{N+p+1;m} \le \lambda_{N+p;m-1}$. Note that by Definition 33,

$$\Lambda_{N+p;m,n} = \ell([0,\lambda_{N+p;m}]\cap(\alpha_n,\alpha_{n-1}]) \le \ell([0,\lambda_{N+p+1;m}]\cap(\alpha_n,\alpha_{n-1}]) = \Lambda_{N+p+1;m,n}. \tag{5.33}$$

Coupling (5.33) with Definition 37, we see that

$$\mathrm{DS}(\Lambda_{N+p})_j = \sum_{k=1}^{M-j+1} \Lambda_{N+p;j-1+k,k} \leq \sum_{k=1}^{M-j+1} \Lambda_{N+p+1;j-1+k,k} = \mathrm{DS}(\Lambda_{N+p+1})_j,$$

as claimed. Indeed, having $\mathrm{DS}(\Lambda_{N+p})_j \leq \mathrm{DS}(\Lambda_{N+p+1})_j$, (5.32) becomes

$$\sum_{p'=1}^{p} \mu_{N+p'} = \sum_{j=1}^{\min\{p,M\}} \mathrm{DS}(\Lambda_{N+p})_j \leq \sum_{j=1}^{\min\{p,M\}} \mathrm{DS}(\Lambda_{N+P})_j = \sum_{j=1}^{\min\{p,M\}} \mathrm{DS}(\Lambda)_j,$$

which is the final majorization condition (5.25) for $p = 1, \ldots, P-1$. □

5.3 Parametrizing continued eigensteps

In the final section of this chapter, we characterize sequences $\{\lambda_m\}_{m=1}^M$ which are (α, μ)-constructible. As discussed in the beginning of the previous section, solving this problem involves finding a valid sequence of continued outer eigensteps for given nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^M$, $\alpha = \{\alpha_m\}_{m=1}^M$, and $\mu = \{\mu_{N+p}\}_{p=1}^P$. Having just shown the necessity of majorization in order for a sequence $\{\lambda_m\}_{m=1}^M$ to be (α, μ)-constructible, in this section, we prove the converse result, namely that if $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^M$ majorizes $\{\mu_{N+p}\}_{p=1}^P$, then a corresponding sequence of continued outer eigensteps $\{\{\lambda_{N+p;m}\}_{m=1}^M\}_{p=0}^P$ exists.

Recall from Chapter 4 the simpler case when no initial spectrum is given, that is, $\alpha_m = 0$ for all m. There we showed the sufficiency of majorization using the Top Kill algorithm. Top Kill removed as much area as possible from the top levels of the staircase (see Figure 4.1) subject to the interlacing and trace constraints. In a similar fashion we develop an algorithm that will show that majorization is sufficient for a sequence $\{\lambda_m\}_{m=1}^M$ to be (α, μ)-constructible; however, this time we remove as much area as possible from the outermost *diagonals*. We now introduce this generalization of Top Kill, called *Chop Kill*. This algorithm gives an explicit construction of a feasible sequence of continued outer eigensteps whenever $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^M$ majorizes $\{\mu_{N+p}\}_{p=1}^P$.

Theorem 40. *Given* $\{\lambda_{N+p;m}\}_{m=1}^M \in \mathrm{adm}(\alpha)$ *and for which* $\{\mathrm{DS}(\Lambda_{N+p})_j\}_{j=1}^M \succeq \{\mu_{N+p'}\}_{p'=1}^p$ *the following Chop Kill algorithm constructs* $\{\lambda_{N+p-1;m}\}_{m=1}^M \in \mathrm{adm}(\alpha)$ *such that* $\{\lambda_{N+p-1;m}\}_{m=1}^M \sqsubseteq \{\lambda_{N+p;m}\}_{m=1}^M$ *and* $\{\mathrm{DS}(\Lambda_{N+p-1})_m\}_{m=1}^M \succeq \{\mu_{N+p'}\}_{p'=1}^{p-1}$*:*

Let $\Lambda_{N+p-1,m,n} = \alpha_{n-1} - \alpha_n$ *for* $n > m$
$\nu_{M,1} := \mu_{N+p}$
for $j = M, \dots, 1$
    for $k = M - j + 1, \dots, 1$
        $\delta = \min\{\nu_{j-1+k,k}, \Lambda_{N+p;j-1+k,k} - \Lambda_{N+p;j+k,k}\}$
        $\Lambda_{N+p-1,j-1+k,k} := \Lambda_{N+p;j-1+k,k} - \delta$
        if $k > 1$
            $\nu_{j-2+k,k-1} := \nu_{j-1+k,k} - \delta$
        else
            $\nu_{M,M-j+2} := \nu_{j-1+k,k} - \delta$
        end
    end
end
$\lambda_{N+p-1;m}$=$[\mathrm{S}(\Lambda_{N+p-1})]_m$ *for all* m

Furthermore, given any nonnegative nonincreasing sequences $\{\alpha_m\}_{m=1}^M$, $\{\lambda_m\}_{m=1}^M$ *and* $\{\mu_{N+p}\}_{p=1}^P$ *such that* $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^M \succeq \{\mu_{N+p}\}_{p=1}^P$*, define* $\lambda_{N+P;m} := \lambda_m$ *for every* $m = 1 \dots, M$*, and for each* $p = P, \dots, 2$*, consecutively define* $\{\lambda_{N+p-1;m}\}_{m=1}^M$ *according to Chop Kill. Then* $\{\{\lambda_{N+p;m}\}_{m=1}^M\}_{p=0}^P$ *are continued outer eigensteps.*

In order to prove Theorem 40, we will need the following lemma:

Lemma 41. *Given* $\{\lambda_{N+p;m}\}_{m=1}^M \in \mathrm{adm}(\alpha)$ *and for which* $\{\mathrm{DS}(\Lambda_{N+p})_j\}_{j=1}^M \succeq \{\mu_{N+p'}\}_{p'=1}^p$*, if* $\{\lambda_{N+p-1;m}\}_{m=1}^M \in \mathrm{adm}(\alpha)$ *is constructed according to the Chop Kill algorithm of Theorem 40, then*

$$\Lambda_{N+p;m+1,n} \leq \Lambda_{N+p-1;m,n} \leq \Lambda_{N+p;m,n}, \tag{5.34}$$

for all $m = 1, \ldots, M$ *and* $n = 1, \ldots, M+1$. *Moreover, there exists an index* $(j_0, k_0) \in \mathcal{J}$ *such that*

$$\Lambda_{N+p-1;j-1+k,k} = \begin{cases} \Lambda_{N+p;j+k,k} & \text{if } (j,k) > (j_0,k_0), \\ \Lambda_{N+p-1;j-1+k,k} & \text{if } (j,k) < (j_0,k_0), \end{cases} \tag{5.35}$$

where "$(j,k) \geq (j_0,k_0)$*" means either* $j > j_0$ *or if* $j = j_0$ *then* $k > k_0$. *Moreover, in the case where* $(j,k) = (j_0,k_0)$*:*

$$\Lambda_{N+p-1;j_0-1+k_0,k_0} = -\mu_{N+p} + \mathrm{DS}(\Lambda_{N+p})_{j_0+1} + \sum_{k=k_0}^{M-j_0+1} \Lambda_{N+p;j_0-1+k,k} - \sum_{k=k_0+1}^{M-j_0+1} \Lambda_{N+p;j_0+k,k}. \tag{5.36}$$

Before proving Theorem 40 and Lemma 41, we give an example in order to demonstrate how the Chop Kill algorithm works.

Example 42. We use the Chop Kill Algorithm of Theorem 40 in order to construct a sequence of continued outer eigensteps for a frame obtained by adding $P = 4$ additional vectors to a set of preexisting vectors in $\mathbb{R}^3$. For this example, our initial starting sequence $\alpha = \{\alpha_m\}_{m=1}^3$ is given by the final sequence of eigenvalues in Example 15 of Chapter 4. Specifically, we let $\{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$. The desired final spectrum is $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$ with the additional vector lengths given by $\{\mu_{N+1}, \mu_{N+2}, \mu_{N+3}, \mu_{N+4}\} = \{2, 1, \frac{1}{4}, \frac{1}{4}\}$. Our goal is to complete the following table according to the rules of continued outer eigensteps (Definition 24):

$N+p$	0	1	2	3	4
$\lambda_{n;3}$	$\frac{1}{2}$	?	?	?	1
$\lambda_{n;2}$	$\frac{3}{4}$	?	?	?	$\frac{9}{4}$
$\lambda_{n;1}$	$\frac{7}{4}$	?	?	?	$\frac{13}{4}$

(5.37)

To be precise, we must pick $\{\lambda_{N+1;m}\}_{m=1}^3$, $\{\lambda_{N+2;m}\}_{m=1}^3$ and $\{\lambda_{N+3;m}\}_{m=1}^3$ that satisfy the interlacing conditions,

$$\{\tfrac{7}{4}, \tfrac{3}{4}, \tfrac{1}{2}\} \sqsubseteq \{\lambda_{N+1;m}\}_{m=1}^3 \sqsubseteq \{\lambda_{N+2;m}\}_{m=1}^3 \sqsubseteq \{\lambda_{N+3;m}\}_{m=1}^3 \sqsubseteq \{\tfrac{13}{4}, \tfrac{9}{4}, 1\},$$

as well as the trace conditions,

$$\lambda_{N+1;1} + \lambda_{N+1;2} + \lambda_{N+1;3} = \sum_{m=1}^{3} \alpha_m + \mu_{N+1} = 5,$$
$$\lambda_{N+2;1} + \lambda_{N+2;2} + \lambda_{N+2;3} = \sum_{m=1}^{3} \alpha_m + \sum_{p'=1}^{2} \mu_{N+p'} = 6,$$
$$\lambda_{N+3;1} + \lambda_{N+3;2} + \lambda_{N+3;3} = \sum_{m=1}^{3} \alpha_m + \sum_{p'=1}^{3} \mu_{N+p'} = \tfrac{25}{4},$$
$$\lambda_{N+4;1} + \lambda_{N+4;2} + \lambda_{N+4;3} = \sum_{m=1}^{3} \alpha_m + \sum_{p'=1}^{4} \mu_{N+p'} = \tfrac{13}{2}.$$

The trace condition means the sum of the values in the $N + p$ column is $\sum_{m=1}^{3} \alpha_m + \sum_{p'=1}^{p} \mu_{N+p'}$, while the interlacing condition means that any value $\lambda_{N+p;m}$ is at least the neighbor to the upper right $\lambda_{N+p+1;m+1}$ and no more than its neighbor to the right, $\lambda_{N+p+1;m}$.

We can view the task of completing (5.37) as iteratively building a staircase. Our goal is to build on top of an existing three-step staircase with steps of length $\{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$ (Figure 5.1(a)) so that the resulting bottom level has length $\frac{13}{4}$, the second level has length $\frac{9}{4}$, and the top level has length 1 (Figure 5.1(b)); the initial and final profile of each staircase is outlined in black in each of the subfigures of Figure 5.1. Similar to Top Kill in Chapter 4, Chop Kill works backwards chipping away at the three-level staircase in Figure 5.1(b) until all of the light gray area is removed. To determine which areas should be removed first, we chop up the staircase into blocks according to the initial spectrum α (indicated by the dotted lines in Figure 5.1(c)), and then place labels on each gray block corresponding to its position relative to the initial spectrum α. Blocks with label "1" in Figure 5.1(d) are one step above the initial spectrum α while blocks with label "2" are two steps above the initial spectrum. As you can see in Figure 5.1(d), areas with the same label form a diagonal along the profile of the staircase. Chop Kill works by removing diagonal "2" first followed by the diagonal labeled "1". Chop Kill derives its name from this process of chopping up the staircase according to α and then "killing" off as much as possible from the outermost diagonals.

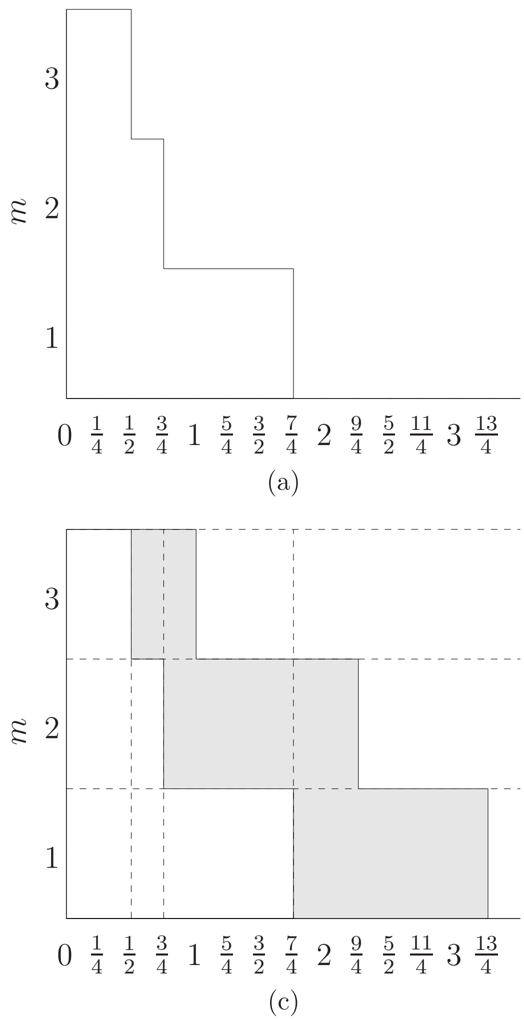

Figure 5.1: Given existing spectrum $\alpha = \{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$ (a), we add 4 additional vectors whose lengths are given by $\{\mu_{N+1}, \mu_{N+2}, \mu_{N+3}, \mu_{N+4}\} = \{2, 1, \frac{1}{4}, \frac{1}{4}\}$ so that the completed frame has spectrum $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$ (b). Our goal is to build a sequence of continued outer eigensteps for the gray area in (b). To determine the priority in which area should be removed, we chop up the staircase into blocks according to the initial spectrum α (indicated by the dotted lines in (c)), and then place labels on each gray block corresponding to its position relative to the initial spectrum α (d). Blocks with label "1" in (d) are one step above the initial spectrum α while blocks with label "2" are two steps above the initial spectrum.

We now attempt to build this staircase, working backwards until all of the gray area has been removed. As just mentioned, our strategy for removal is to remove diagonal "2" first,

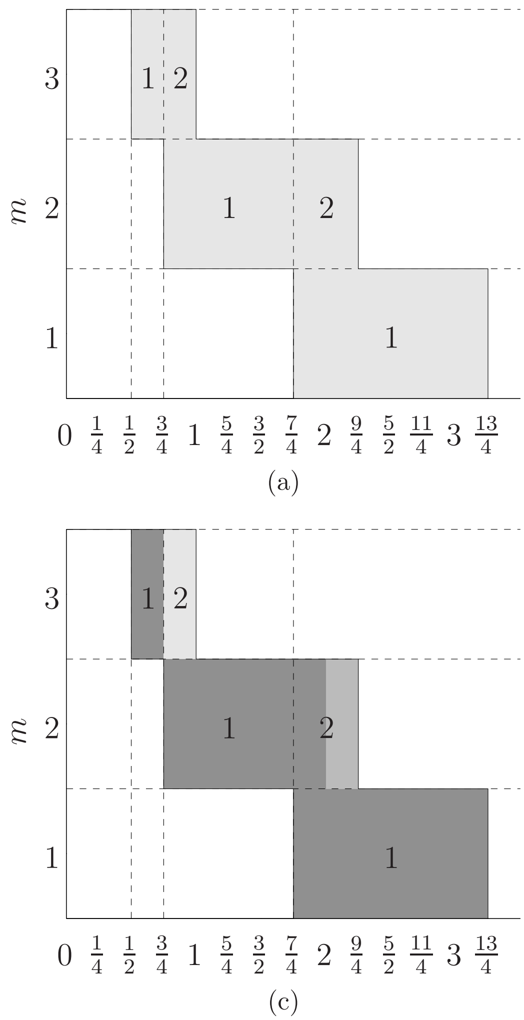

Figure 5.2: Iteratively building a sequence of continued outer eigensteps for $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$ where $\alpha = \{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$ and $\{\mu_{N+1}, \mu_{N+2}, \mu_{N+3}, \mu_{N+4}\} = \{2, 1, \frac{1}{4}, \frac{1}{4}\}$. Beginning with the final desired staircase, we work backwards to the initial staircase. That is, we chip away at the three-level staircase (a) to produce (b), chip away at (b) to produce (c), and then finally chip away from (c) to produce (d). In each step, we do this by removing as much as possible from the top diagonals before turning our attention to the lower diagonals, subject to the interlacing and trace constraints. We refer to this algorithm for iteratively producing $\{\lambda_{N+p-1;m}\}_{m=1}^{M}$ from $\{\lambda_{N+p;m}\}_{m=1}^{M}$ as Chop Kill.

followed by diagonal "1". We begin with the final desired spectrum $\{\lambda_{N+4;1}, \lambda_{N+4;2}, \lambda_{N+4;3}\} = \{\frac{13}{4}, \frac{9}{4}, 1\}$ (Figure 5.2(a)). Observe in Figure 5.2(a), that the highest portion of diagonal "2" is at the third level of the staircase. As such, we remove $\mu_{N+4} = \frac{1}{4}$ units of area from

this level resulting in a staircase—the darker shade in Figure 5.2(b)—which has levels of lengths $\{\lambda_{N+3;1}, \lambda_{N+3;2}, \lambda_{N+3;3}\} = \{\frac{13}{4}, \frac{9}{4}, \frac{3}{4}\}$. In the next step we remove a block of area $\mu_{N+3} = \frac{1}{4}$ from the second level of the staircase. We remove from the second level since it contains the highest instance of diagonal "2" that has yet to be removed. This results in a staircase with lengths $\{\lambda_{N+2;1}, \lambda_{N+2;2}, \lambda_{N+2;3}\} = \{\frac{13}{4}, 2, \frac{3}{4}\}$–again, the darker shade in Figure 5.2(c). In the third step, we apply this same philosophy, this time removing two blocks of area from the second and third levels whose total area will be $\mu_{N+2} = 1$. Since there is still part of "2" left at the second level, we chip away this remaining $\frac{1}{4}$ units of area first. At this point, all of the "2" area has been removed, which leaves $\frac{3}{4}$ units of area to remove from diagonal "1". Accordingly, we chip away $\frac{1}{4}$ units of area from level three and an additional $\frac{1}{2}$ units of area from level two resulting in a staircase with levels $\{\lambda_{N+1;1}, \lambda_{N+1;2}, \lambda_{N+1;3}\} = \{\frac{13}{4}, \frac{5}{4}, \frac{1}{2}\}$ (Figure 5.2(d)). In the final step, we remove the remaining areas of diagonal "1" totaling $\mu_{N+1} = 2$ units of area. This process produce the following valid sequence of continued outer eigensteps:

$$\begin{array}{c|ccccc} N+p & 0 & 1 & 2 & 3 & 4 \\ \hline \lambda_{n;3} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & 1 \\ \lambda_{n;2} & \frac{3}{4} & \frac{5}{4} & 2 & \frac{9}{4} & \frac{9}{4} \\ \lambda_{n;1} & \frac{7}{4} & \frac{13}{4} & \frac{13}{4} & \frac{13}{4} & \frac{13}{4} \end{array} \tag{5.38}$$

In the analysis that follows, we show that the Chop Kill algorithm does indeed produce this same sequence of continued outer eigensteps.

Now that we know intuitively how the Chop Kill algorithm works, we repeat the process of constructing (5.38) using the algorithm given in Theorem 40. We will apply Chop Kill three times, once for each of the undefined columns in (5.37). As with Top Kill, we work backwards, producing the third column from the fourth, then the second from the third, and so on. For each column, the Chop Kill algorithm first finds the spectral partition matrix for the eigenvalues belonging in that particular column (Lines 01-13). For this

example, $M = 3$, so each spectral partition matrix that is constructed will be of size 3×4. Chop Kill defines the entries of each spectral partition matrix one at a time—sweeping through each lower diagonal from right to left. As a final step (Line 14), Chop Kill sums across the rows of the spectral partition matrix produced by Lines 01-13 to obtain the desired eigenvalues.

We begin with $p = 4$, and use the Chop Kill algorithm to construct Λ_{N+3}, the spectral partition matrix for $\{\lambda_{N+3;m}\}_{m=1}^{3}$. In order to do so, we need to know Λ_{N+4}, since the value of δ (Line 05) depends in part on a differences between matrix entries of Λ_{N+4}. For the values of $\{\lambda_m\}_{m=1}^{3}$ chosen in this example, we have already shown in Example 34 how to construct this spectral partition matrix using the function P_α (Definition 33):

$$\Lambda_{N+4} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

At Line 01, we begin by setting $\Lambda_{N+3;m,n} = \alpha_{n-1} - \alpha_n$ for $n > m$. In essence, this is done to ensure that the resulting matrix Λ_{N+3}—which much be α-admissible—satisfies Definition 35.(iii). This completes the upper triangular portion of Λ_{N+3} as follows:

$$\Lambda_{N+3} = \begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ ? & ? & \frac{1}{4} & \frac{1}{2} \\ ? & ? & ? & \frac{1}{2} \end{bmatrix}$$

Next at Line 02, we set $\nu_{3,1} = \mu_{N+4} = \frac{1}{4}$. Since we are removing a total area of μ_{N+4}, these "ν" quantities track how much area is left to be removed at future iterations of the "for-loops" (Lines 03-13). The for-loops set the values of the diagonals of Λ_{N+3} working from the outermost diagonal to the main diagonal, and then from right to left within each diagonal. Referring to the matrix indexing given in (5.22), the Chop Kill algorithm sets the values of Λ_{N+3} in the following order:

$$(3, 1),\ (3, 2),\ (2, 1),\ (3, 3),\ (2, 2),\ (1, 1).$$

Beginning with $j = 3$ (Line 03), the outermost diagonal only consists of one element and so $k = 1$. At Line 05, we set

$$\delta = \min\{\nu_{3,1}, \Lambda_{N+4;3,1} - \Lambda_{N+4;4,1}\} = \min\{\tfrac{1}{4}, 0\} = 0,$$

where $\Lambda_{N+4;4,:} := 0$, which gives $\Lambda_{N+3;3,1} = \Lambda_{N+4;3,1} - \delta = 0 - 0 = 0$ (Line 06). The next value of ν is defined by Lines 07-11 depending on the value of k, and since $k = 1$, $\nu_{3,2} := \nu_{3,1} - \delta = \frac{1}{4} - 0 = \frac{1}{4}$. At the end of the for-loop for $j = 3$, we have:

$$\Lambda_{N+3} = \begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ ? & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & ? & ? & \frac{1}{2} \end{bmatrix}$$

Next for $j = 2$, the Chop Kill algorithm sweeps from right to left along the 2nd lower diagonal of Λ_{N+3}. The 2nd diagonal consists of two elements and so k ranges from 1 to 2. Sweeping from right to left along this diagonal is equivalent to working backwards from $k = 2, \ldots, 1$. When $k = 2$, Line 05 gives

$$\delta = \min\{\nu_{3,2}, \Lambda_{N+4;3,2} - \Lambda_{N+4;4,2}\} = \min\{\tfrac{1}{4}, 0\} = \tfrac{1}{4},$$

which means that $\frac{1}{4}$ units of area will be removed from $\Lambda_{N+4;3,2}$, i.e., $\Lambda_{N+3;3,2} = \Lambda_{N+4;3,2} - \delta = \frac{1}{4} - \frac{1}{4} = 0$. Since $\frac{1}{4}$ unit of area is removed and $\mu_{N+4} = \frac{1}{4}$, we expect the next value of ν to be zero since all of the area has already been removed; this indeed is true:

$$\nu_{2,1} = \nu_{3,2} - \delta = \tfrac{1}{4} - \tfrac{1}{4} = 0.$$

In fact, once the value of ν becomes zero, all remaining values of ν computed in subsequent iterations, particularly $\nu_{3,3}$, $\nu_{2,2}$, and $\nu_{1,1}$, will be zero as well. As such, δ (Line 05) will also become zero. This simplifies the calculations for the remainder of the iterations and Line 06 will simplify to $\Lambda_{N+p-1,j-1+k,k} := \Lambda_{N+p;j-1+k,k}$. Notice in all the calculations up to this point, $\delta = \Lambda_{N+p;j-1+k,k} - \Lambda_{N+p;j+k,k}$, giving that $\Lambda_{N+p-1,j-1+k,k} := \Lambda_{N+p;j+k,k}$. Using this fact,

we expect Λ_{N+3} to be

$$\Lambda_{N+3} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \tag{5.39}$$

Later we will show that this "transition" point $(j_0, k_0) = (3, 2) \in \mathcal{J}$ is unique and that all the matrix entries of Λ_{N+p-1} can be defined according to this unique index pair. We formalize this result in Lemma 41 and use it to prove Theorem 40. While knowing this transition index greatly simplifies our calculations, its value cannot be determined explicitly in advance, and so, the Chop Kill algorithm is still the main tool we use in order to construct a sequence of continued outer eigensteps.

To verify that (5.39) is indeed true, we return to our example and pick back up with the case when $j = 2$ and $k = 1$. In this case,

$$\delta = \min\{\nu_{2,1}, \Lambda_{N+4;2,1} - \Lambda_{N+4;3,1}\} = \min\{0, \tfrac{1}{2} - 0\} = 0,$$

which is what we expected since all of μ_{N+4} was removed in the previous iteration. Line 06 gives that $\Lambda_{N+3;2,1} := \Lambda_{N+4;2,1} = \frac{1}{2}$ and at Line 08, we have $\nu_{3,3} = 0$, which finishes the 2nd diagonal of Λ_{N+3}:

$$\Lambda_{N+3} = \begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & ? & \frac{1}{2} \end{bmatrix}$$

What remains is the main diagonal which corresponds to the case when $j = 1$. Since this diagonal has three elements, the Chop Kill algorithm works backwards from

$k = 3, \ldots, 1$. We summarize these calculations in the following table:

$$\begin{array}{ccccc} j & k & \delta & \Lambda_{N+3,j-1+k,k} & \nu & \Lambda_{N+3} \\ \hline 1 & 3 & 0 & \Lambda_{N+3,3,3} := \frac{1}{4} & \nu_{2,2} := 0 & \begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \\ 1 & 2 & 0 & \Lambda_{N+3,2,2} := 1 & \nu_{1,1} := 0 & \begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \\ 1 & 1 & 0 & \Lambda_{N+3,1,1} := \frac{3}{2} & \nu_{3,4} := 0 & \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array} \tag{5.40}$$

In each of these last few steps, the new value in the Λ_{N+3} matrix is set to the value in the corresponding location of Λ_{N+4}. As we expected, all values of δ and ν in columns three and five of (5.40) are set to zero since all of μ_{N+4} has already been removed. At Line 13 of the Chop Kill algorithm, we have indeed produced (5.39), the spectral partition matrix Λ_{N+3}. Having this spectral partition matrix, we now sum across its rows (Line 14), which gives the fourth column of (5.38).

Next we repeat the Chop Kill algorithm for $p = 3$ and construct Λ_{N+2} by removing a total of $\mu_{N+3} = \frac{1}{4}$ units of area. Again we work from the outermost diagonal and from right to left along each diagonal. For the outermost diagonal, $j = 3$, the Chop Kill algorithm produces:

$$\begin{array}{cccccc} j & k & \delta & \Lambda_{N+2,j-1+k,k} & \nu & \Lambda_{N+2} \\ \hline 3 & 1 & 0 & \Lambda_{N+2,3,1} := 0 & \nu_{3,2} := \frac{1}{4} & \begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ ? & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & ? & ? & \frac{1}{2} \end{bmatrix} \end{array}$$

For the next iteration $j = 2$, the Chop Kill algorithm defines the second diagonal as follows:

j	k	δ	$\Lambda_{N+2,j-1+k,k}$	ν	Λ_{N+2}
2	2	0	$\Lambda_{N+2,3,2} := 0$	$\nu_{2,1} := \frac{1}{4}$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ ? & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & ? & \frac{1}{2} \end{bmatrix}$
2	1	$\frac{1}{4}$	$\Lambda_{N+2,2,1} := \frac{1}{4}$	$\nu_{3,3} := 0$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & ? & \frac{1}{2} \end{bmatrix}$

Notice in these calculations when $j = 2$ and $k = 1$, $\frac{1}{4}$ units of area are removed from the $(2, 1)$ entry of Λ_{N+3}. Since all of μ_{N+3} is removed at this iteration, the next value of ν, $\nu_{3,3}$, is set to zero. In this case, the unique index (j_0, k_0) which corresponds to when all of μ_{N+3} has been removed is $(2, 1)$. Finally, for the last iteration when $j = 1$:

j	k	δ	$\Lambda_{N+2,j-1+k,k}$	ν	Λ_{N+2}
1	3	0	$\Lambda_{N+2,3,3} := \frac{1}{4}$	$\nu_{2,2} := 0$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$
1	2	0	$\Lambda_{N+2,2,2} := 1$	$\nu_{1,1} := 0$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$
1	1	0	$\Lambda_{N+2,1,1} := \frac{3}{2}$	$\nu_{3,4} := 0$	$\begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

This yields the resulting matrix for Λ_{N+2}:

$$\Lambda_{N+2} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

Summing across the rows of Λ_{N+2} (Line 14) yields the third column of (5.38) as desired.

Finally, we repeat this process for $p = 2$, removing $\mu_{N+2} = 1$ in order to construct Λ_{N+1}. The evolution of Λ_{N+1} at each iteration of the Chop Kill algorithm is summarized in the following table:

j	k	δ	$\Lambda_{N+1,j-1+k,k}$	ν	Λ_{N+1}
3	1	0	$\Lambda_{N+1,3,1} := 0$	$\nu_{3,2} := 1$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ ? & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & ? & ? & \frac{1}{2} \end{bmatrix}$
2	2	0	$\Lambda_{N+2,3,2} := 0$	$\nu_{2,1} := 1$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ ? & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & ? & \frac{1}{2} \end{bmatrix}$
2	1	$\frac{1}{4}$	$\Lambda_{N+2,2,1} := 0$	$\nu_{3,3} := \frac{3}{4}$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & ? & \frac{1}{2} \end{bmatrix}$
1	3	$\frac{1}{4}$	$\Lambda_{N+2,3,3} := 0$	$\nu_{2,2} := \frac{1}{2}$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & ? & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$
1	2	$\frac{1}{2}$	$\Lambda_{N+2,2,2} := \frac{1}{2}$	$\nu_{1,1} := 0$	$\begin{bmatrix} ? & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$
1	1	0	$\Lambda_{N+2,1,1} := \frac{3}{2}$	$\nu_{3,4} := 0$	$\begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

Summing across the rows of the resulting matrix Λ_{N+1} does in fact produce the eigensteps in the second column of (5.38).

For $p = 1$, the Chop Kill algorithm produces Λ_N which is already given by the initial sequence of eigenvalues α. It is straightforward to verify that the Chop Kill algorithm produces the same Λ_N that can found by applying P_α, given by (5.13), to α. While these calculations seem very tedious, it is very easy automated using MATLAB. The code can be found in the appendix.

Note that one application of Chop Kill generates $\{\lambda_{N+p-1;m}\}_{m=1}^{M}$ from $\{\lambda_{N+p;m}\}_{m=1}^{M}$. Just as demonstrated in the example, one may start with any $\{\lambda_{N+P;m}\}_{m=1}^{M} := \{\lambda_m\}_{m=1}^{M}$ and apply the Chop Kill algorithm $P-1$ times in order to produce a valid sequence of continued outer eigensteps $\{\{\lambda_{N+p;m}\}_{m=1}^{M}\}_{p=0}^{P}$ that satisfies Definition 24.

Having discussed how Chop Kill works, we now prove Lemma 41 first followed by the proof of Theorem 40.

Proof of Lemma 41. For the sake of notational simplicity, we denote $B := \Lambda_{N+p-1}$ and $C := \Lambda_{N+p}$ meaning $B_{j,k} := \Lambda_{N+p-1;j,k}$ and $C_{j,k} := \Lambda_{N+p;j,k}$. We begin by proving (5.34). Specifically we wish to show that

$$C_{m+1,n} \leq B_{m,n} \leq C_{m,n}. \tag{5.41}$$

To do so, we consider the case when $n > m$ and the case when $n \leq m$. In the case that $n > m$, the fact that $C \in \mathrm{Adm}(\alpha)$ implies

$$C_{m+1,n} \leq C_{m,n} = \alpha_{n-1} - \alpha_n. \tag{5.42}$$

From Line 01 of the Chop Kill algorithm, $B_{m,n} = \alpha_{n-1} - \alpha_n$; substituting this into (5.42) gives

$$C_{m+1,n} \leq C_{m,n} = \alpha_{n-1} - \alpha_n = B_{m,n} = \alpha_{n-1} - \alpha_n = C_{m,n},$$

and so (5.41) holds in the case that $n > m$. On the other hand if $n \leq m$, note that Line 05 gives $\delta \leq C_{j-1+k,k} - C_{j+k,k}$. Substituting this for δ at Line 06 gives

$$B_{j-1+k,k} \geq C_{j-1+k,k} - (C_{j-1+k;k} - C_{j+k;k}) = C_{j+k;k}. \tag{5.43}$$

We also have that $\delta \leq \nu_{j-1+k,k}$ which follows from Line 05 as well. This implies that all new values of ν defined at Lines 08 and 10 are nonnegative; specifically, $\nu_{j-2+k,k} \geq 0$ and $\nu_{M,M-j+2} \geq 0$ for all $j \leq k$. Combining this with the fact that $C_{m,n} - C_{m-1+n,n} \geq 0$—due to the fact that $C \in \mathrm{Adm}(\alpha)$—implies $\delta \geq 0$ (Line 05). As such, Line 06 gives

$$B_{j-1+k,k} = C_{j-1+k,k} - \delta \leq C_{j-1+k;k}. \tag{5.44}$$

Combining (5.43) and (5.44) and making a change of variables according to (5.20), we have that (5.41) holds in the case that $n \leq m$.

Next we prove (5.35). First, we place an ordering on the set of all pairs of indices $\mathcal{J}$. Specifically we write $(j,k) \geq (j',k')$ if either $j > j'$ or if $j = j'$ and $k > k'$. Notice that the algorithm computes the entries of B from those of C in this order working from greatest to least. That is, at first it defines values of B at $(M,1)$, then at $(M,2)$, $(M-1,1)$, etc., finishing at $(1,1)$. We now let (j_0,k_0) be the maximal pair (j,k) such that

$$\nu_{j-1+k,k} \leq C_{j-k+1,k} - C_{j-k,k}. \tag{5.45}$$

To show that (j_0,k_0) is well-defined, we must show that (5.45) holds for at least one $(j,k) \in \mathcal{J}$. Suppose to the contrary that there is no such pair (j,k) such that (5.45) holds, That is,

$$\nu_{j-1+k,k} > C_{j-1+k,k} - C_{j+k,k}. \tag{5.46}$$

In particular for $(j,k) = (1,1)$

$$\nu_{1,1} > C_{1,1} - C_{2,1}. \tag{5.47}$$

Moreover, for $(j,k) > (1,1)$, (5.46) implies that δ of Line 05 is given by

$$\delta = C_{j-1+k,k} - C_{j+k,k}.$$

As such, progressing through all but the last step of the Chop Kill algorithm, namely progressing from $(j,k) = (M,1)$ down to $(j,k) = (1,2)$, gives

$$\nu_{1,1} := \mu_{N+p} - \sum_{j=2}^{M} \sum_{k=1}^{M-j+1} (C_{j-1+k,k} - C_{j+k,k}) - \sum_{k=2}^{M} (C_{k,k} - C_{k+1,k}). \tag{5.48}$$

Combining (5.47) with (5.48) and rearranging gives:

$$\mu_{N+p} > \sum_{j=1}^{M} \sum_{k=1}^{M-j+1} (C_{j-1+k,k} - C_{j+k,k}). \tag{5.49}$$

Next we make a change of variables according to (5.20). This allows us to rewrite (5.49) as follows:

$$\mu_{N+p} > \sum_{m=1}^{M} \sum_{n=1}^{m} (C_{m,n} - C_{m+1,n}) = \sum_{m=1}^{M} \sum_{n=1}^{m} C_{m,n} - \sum_{m=2}^{M+1} \sum_{n=1}^{m-1} C_{m,n}. \tag{5.50}$$

Since $C_{M+1,n} := 0$, (5.50) further reduces to

$$\begin{aligned}
\mu_{N+p} &> \sum_{m=1}^{M} \sum_{n=1}^{m} C_{m,n} - \sum_{m=2}^{M} \sum_{n=1}^{m-1} C_{m,n} \\
&= C_{1,1} + \sum_{m=2}^{M} \sum_{n=1}^{m} C_{m,n} - \sum_{m=2}^{M} \sum_{n=1}^{m-1} C_{m,n} \\
&= C_{1,1} + \sum_{m=2}^{M} C_{m,m} \\
&= \mathrm{DS}(C)_1.
\end{aligned}$$

This contradicts our assumption that $\{\mathrm{DS}(C)_j\}_{j=1}^{M} \geq \{\mu_{N+p}\}_{p=1}^{P}$—specifically this contradicts the fact that $\mathrm{DS}(C)_1 \geq \mu_{N+1} \geq \mu_{N+p}$. As such, (5.45) holds for at least one pair (j, k) and so (j_0, k_0) is well-defined.

We now prove (5.35), that is, for all but the (j_0, k_0)th entry, the "new" spectral partition matrix entry is either a copy of the existing entry in that spot or a copy of the entry directly below it. To see this fact, note that if $(j, k) > (j_0, k_0)$, then since (j_0, k_0) is the maximal pair such that (5.46) holds, then (5.46) does not hold for our particular choice of (j, k), implying δ at Line 05 is $\delta = C_{j-1+k,k} - C_{j+k,k}$. At this point, Line 06 gives

$$B_{j-1+k,k} = C_{j-1+k,k} - (C_{j-1+k,k} - C_{j+k,k}) = C_{j+k,k},$$

namely (5.35) in the case that $(j, k) > (j_0, k_0)$. To prove (5.35) holds in the remaining case where $(j, k) < (j_0, k_0)$, note that at the (j_0, k_0)th step of the Chop Kill algorithm, the fact

that (5.45) holds implies $\delta = \nu_{j-1+k,k}$. Thus subsequent values of ν defined at Lines 08 and 10, namely $\nu_{j-2+k,k}$ when $k > 1$ and $\nu_{M,M-j+2}$ when $k = 1$, will be zero. As such, for all remaining steps of the Chop Kill algorithm (namely those where $(j,k) < (j_0,k_0)$), Line 05 gives $\delta = 0$ at which point Line 06 gives (5.35) in the case that $(j,k) < (j_0,k_0)$.

Finally, we must show that (5.36) holds in the case that $(j,k) = (j_0,k_0)$. Note that since (j_0,k_0) is the *maximal* index such that (5.45) holds, at Line 06 of the Chop Kill algorithm,

$$B_{j_0-1+k_0,k_0} = C_{j_0,-1+k_0,k_0} - \delta = C_{j_0-1+k_0,k_0} - \nu_{j_0-1+k_0,k_0} \tag{5.51}$$

where $\nu_{j_0-1+k_0,k_0}$ is given by

$$\nu_{j_0-1+k_0,k_0} = \mu_{N+p} - \sum_{j=j_0+1}^{M} \sum_{k=1}^{M-j+1} (C_{j-1+k,k} - C_{j+k,k}) - \sum_{k=k_0+1}^{M-j_0+1} (C_{j_0-1+k,k} - C_{j_0+k,k}). \tag{5.52}$$

Substituting (5.52) into (5.51) gives (5.36) as claimed. □

Proof of Theorem 40. For the sake of notational simplicity, we denote $\{\beta_m\}_{m=1}^M := \{\lambda_{N+p-1;m}\}_{m=1}^M$, $B_{j,k} := \Lambda_{N+p-1;j,k}$, $B := \Lambda_{N+p-1}$, and denote $\{\gamma_m\}_{m=1}^M := \{\lambda_{N+p;m}\}_{m=1}^M$, $C_{j,k} := \Lambda_{N+p;j,k}$ and $C := \Lambda_{N+p}$. First we show that $B \in \mathrm{Adm}(\alpha)$ by showing that B satisfies all three properties of Definition 35. Property (iii) follows immediately from Line 01 that $B_{m,n} = \alpha_{n-1} - \alpha_n$ for $n > m$. Property (ii) is satisfied as a result of Lemma 41—in particular from (5.34) . Combining (5.34) at m and (5.34) at $m-1$,

$$C_{m+1,n} \leq B_{m,n} \leq C_{m,n} \leq B_{m-1,n} \leq C_{m-1,n}.$$

Comparing the second and fourth terms gives property (ii) that $B_{m,n} \leq B_{m-1,n}$. In order to show property (i) holds we consider several different cases. To be clear, assume that $B_{m_1,n_1} > 0$ for some (m_1,n_1). We must show that

$$B_{m_1,n} = \alpha_{n-1} - \alpha_n, \tag{5.53}$$

for $n > n_1$. From Line 01 of the Chop Kill algorithm, (5.53) immediately holds for $n > m_1$ and so it suffices to prove (5.53) for $n_1 < n \leq m_1$. In particular since $n_1 < m_1$, the index pair

$(m_1, n_1) \in \mathcal{M}$, the lower triangular part of B. By applying the coordinate transform (5.20), (m_1, n_1) can be written as:

$$(m_1, n_1) = \sigma(j_1, k_1) = (j_1 - 1 + k_1, k_1)$$

for some unique $(j_1, k_1) \in \mathcal{J}$. We now prove (5.53) by separately considering the case where $(j_1, k_1) \leq (j_0, k_0)$ and the case where $(j_1, k_1) > (j_0, k_0)$. If $(j_1, k_1) \leq (j_0, k_0)$ then by Lemma 41, $B_{j-1+k,k} = C_{j-1+k,k}$ for all $(j, k) < (j_1, k_1)$. As such

$$B_{m_1,n} = C_{m_1,n}, \tag{5.54}$$

for all $n > n_1$. Moreover since $(j_1, k_1) \leq (j_0, k_0)$, then the assumption that $B_{j_1-1+k_1,k_1} = B_{m_1,n_1} > 0$ implies $C_{m_1,n_1} \geq B_{m_1,n_1} > 0$ as a result of Lemma 41. Combining this with the fact that $C \in \mathrm{Adm}(\alpha)$ we have

$$C_{m_1,n_1} = \alpha_{n-1} - \alpha_n, \tag{5.55}$$

for all $n > n_1$. Putting (5.54) and (5.55) together gives $B_{m_1,n} = \alpha_{n-1} - \alpha_n$ for all $n > n_1$ as claimed. In the case where $(j_1, k_1) > (j_0, k_0)$, Lemma 41 gives $B_{m_1,n_1} = C_{m_1+1,n_1}$. Note that since $B_{m_1,n_1} > 0$, this immediately rules out $m_1 = M$. Since $C_{m_1+1,n_1} = B_{m_1,n_1} > 0$ where $C \in \mathrm{Adm}(\alpha)$, then

$$C_{m_1+1,n} = \alpha_{n-1} - \alpha_n,$$

for all $n > n_1$. By Lemma 41

$$\alpha_{n-1} - \alpha_n = C_{m_1+1,n} \leq B_{m_1,n} \leq C_{m_1,n} \leq \alpha_{n-1} - \alpha_n$$

where the last inequality follows from $C \in \mathrm{Adm}(\alpha)$, in particular the fact that the rows of C are α-partitions (see Definition 28). Thus $B_{m_1,n} = \alpha_{n-1} - \alpha_n$ for all $n > n_1$ as claimed and so $B \in \mathrm{Adm}(\alpha)$.

Next we show that $\{\beta_m\}_{m=1}^M \sqsubseteq \{\gamma_m\}_{m=1}^M$. Using the results of Lemma 41 and the fact that B and C are both α-admissible, we have

$$\gamma_{m+1} = \sum_{m=1}^{M+1} C_{m+1,n} \leq \sum_{m=1}^{M+1} B_{m,n} = \beta_m, \tag{5.56}$$

$$\beta_m = \sum_{m=1}^{M+1} B_{m,n} \leq \sum_{m=1}^{M+1} C_{m,n} = \gamma_m. \tag{5.57}$$

Putting (5.56) and (5.57) together, $\gamma_{m+1} \leq \beta_m \leq \gamma_m$ and so $\{\beta_m\}_{m=1}^M \sqsubseteq \{\gamma_m\}_{m=1}^M$ as desired.

For the final claim that $\{\mathrm{DS}(B)_j\}_{j=1}^M$ majorizes $\{\mu_n\}_{n=N+1}^{N+p-1}$, first we write $\{\mathrm{DS}(B)_j\}_{j=1}^M$ in terms of $\{\mathrm{DS}(C)_j\}_{j=1}^M$. To do this, we use the result of Lemma 41 that there exists an index $(j_0, k_0) \in \mathcal{J}$ such that (5.35) and (5.36) hold. In the case that $j < j_0$,

$$\mathrm{DS}(B)_j = \sum_{k=1}^{M-j+1} B_{j-1+k,k} = \sum_{k=1}^{M-j+1} C_{j-1+k,k} = \mathrm{DS}(C)_j. \tag{5.58}$$

On the other hand, in the case that $j > j_0$,

$$\begin{aligned}\mathrm{DS}(B)_j &= \sum_{k=1}^{M-j+1} B_{j-1+k,k} = \sum_{k=1}^{M-j+1} C_{j+k,k} \\ &= C_{M+1,M-j+1} + \sum_{k=1}^{M-(j+1)-1} C_{(j+1)-1+k,k} = \mathrm{DS}(C)_{j+1}.\end{aligned} \tag{5.59}$$

since $C_{M+1,n} := 0$. Finally in the case where $j = j_0$, $\mathrm{DS}(B)_{j_0}$ becomes:

$$\sum_{k=1}^{M-j_0-1} B_{j_0-1+k,k} = \sum_{k=1}^{k_0-1} C_{j_0-1+k,k} + B_{j_0-1+k_0,k_0} + \sum_{k=k_0+1}^{M-j_0+1} C_{j_0+k,k}. \tag{5.60}$$

Substituting (5.36) into (5.60),

$$\begin{aligned}\mathrm{DS}(B)_{j_0} &= \sum_{j=j_0+1}^{M} \mathrm{DS}(C)_j - \sum_{j=j_0+1}^{M} \mathrm{DS}(C)_{j+1} + \sum_{k=1}^{M-j_0+1} C_{j_0-1+k,k} - \mu_{N+p} \\ &= \sum_{j=j_0}^{M} \mathrm{DS}(C)_j - \sum_{j=j_0+1}^{M} \mathrm{DS}(C)_{j+1} - \mu_{N+p} \\ &= \mathrm{DS}(C)_{j_0} + \mathrm{DS}(C)_{j_0+1} - \mu_{N+p}.\end{aligned} \tag{5.61}$$

Taking (5.58), (5.59), and (5.61) together, gives the following formula for the diagonals $\{\mathrm{DS}(B)_j\}_{j=1}^{M}$:

$$\mathrm{DS}(B)_j = \begin{cases} \mathrm{DS}(C)_j, & \text{if } j < j_0, \\ \mathrm{DS}(C)_j + \mathrm{DS}(C)_{j+1} - \mu_{N+p}, & \text{if } j = j_0, \\ \mathrm{DS}(C)_{j+1}, & \text{if } j > j_0. \end{cases} \tag{5.62}$$

We next show that $\{\mathrm{DS}(B)_j\}_{j=1}^{M} \geq \{\mu_{N+p'}\}_{p'=1}^{p-1}$. If $l \leq j_0 - 1$, then since $\{\mathrm{DS}(C)_j\}_{j=1}^{M} \geq \{\mu_n\}_{n=N+1}^{N+p}$ we have

$$\sum_{j=1}^{l} \mathrm{DS}(B)_j = \sum_{j=1}^{l} \mathrm{DS}(C)_j \geq \sum_{n=1}^{l} \mu_{N+n}.$$

On the other hand if $l \geq j_0$, we have

$$\begin{aligned} \sum_{j=1}^{l} \mathrm{DS}(B)_j = \sum_{j=1}^{j_0-1} \mathrm{DS}(C)_j + \mathrm{DS}(C)_{j_0} + \mathrm{DS}(C)_{j_0+1} \\ - \mu_{N+p} + \sum_{n=j_0+1}^{l} \mathrm{DS}(C)_{j+1}, \end{aligned} \tag{5.63}$$

with the understanding that a sum over an empty set of indices is zero. We continue (5.63) by using the facts that $\{\mathrm{DS}(C)_j\}_{j=1}^{M} \geq \{\mu_{N+p'}\}_{p'=1}^{p}$ and $\mu_{N+l+1} \geq \mu_{N+p}$:

$$\sum_{j=1}^{l} \mathrm{DS}(B)_j = \sum_{j=1}^{l+1} \mathrm{DS}(C)_j - \mu_{N+p} \geq \sum_{n=1}^{l+1} \mu_{n+N} - \mu_{N+p} \geq \sum_{n=1}^{l} \mu_{n+N}. \tag{5.64}$$

Note that when $l = M$, the inequalities in (5.64) become equalities, giving the final trace condition.

For the final conclusion, note that one application of Chop Kill transforms a sequence $\{\gamma_m\}_{m=1}^{M}$ with the property that $\{\mathrm{DS}(C)_j\}_{j=1}^{M} \geq \{\mu_{N+p'}\}_{p'=1}^{p}$ into a shorter sequence $\{\beta_m\}_{m=1}^{M}$ which also has the property that $\{\mathrm{DS}(B)_j\}_{j=1}^{M} \geq \{\mu_{N+p'}\}_{p'=1}^{p-1}$. As such, one may indeed start with $\lambda_{N+P;m} := \lambda_m$ and apply Chop Kill $P-1$ times to produce a sequence of continued outer eigensteps that immediately satisfies Definition 24. ◻

To summarize the results of this chapter, note we have completely solved the frame completion problem given in Problem 23: by combining results of Theorems 39 and 40, we obtain the following result:

Corollary 43. *Given nonnegative nonincreasing sequences* $\{\alpha_m\}_{m=1}^M$, $\{\mu_{N+p}\}_{p=1}^P$ *and* $\{\lambda_m\}_{m=1}^M$, *which satisfies* $\lambda_m \geq \alpha_m$ *for all* $m = 1, \dots, M$, *then* $\{\lambda_m\}_{m=1}^M$ *is* (α, μ)*-constructible if and only if* $\{\mathrm{DS}(\Lambda)_m\}_{j=1}^M \succeq \{\mu_{N+p}\}_{p=1}^P$.

VI. Optimal frame completions

In this chapter, we build on the theory of Chapter 5 to find the optimal (α,μ)-constructible sequence. The main result of this chapter is Theorem 48 which provides an explicit formula for the optimal (α,μ)-constructible sequence in the special case that $\mu = \{\mu_{N+p}\}_{p=1}^{P}$ are all of equal lengths. In particular, we provide a partial solution to the following problem:

Problem 44. *Given nonnegative nonincreasing sequences $\alpha = \{\alpha_m\}_{m=1}^{M}$ and $\mu = \{\mu_{N+p}\}_{p=1}^{P}$, find the (α,μ)-constructible sequence which is optimal with respect to the MSE, FP, or some other given convex functional of the eigenvalues of the frame operator.*

We are interested in solving Problem 44 because of its real-world applications. For example, if we consider transmitting an encoded signal over a noisy channel, it is possible that part of the signal being transmitted will be distorted or even lost. As such, we desire an encoding scheme that will be as resilient to these errors as possible. It is already well known that optimal frames in such situations are UNTFs as they are minimizers of the MSE and FP [5, 26]. Constructing a UNTF for a given application may not be achievable, however, if given an initial frame, one is restricted to adding only a finite number of new measurements. In such cases, the solution to Problem 44 is not obvious.

To find the optimal (α,μ)-constructible sequence, we follow an approach found in [40] and [41] and measure optimality with respect to majorization. Here it is shown that optimal frame completions are minimizers of a family of convex functionals that includes, but is not limited to, the MSE and FP. Later we will show that the optimal (α,μ)-constructible sequence is the one which is majorized by all other sequences in the following set:

Definition 45. Given nonnegative nonincreasing sequences $\alpha = \{\alpha_m\}_{m=1}^{M}$ and $\mu = \{\mu_{N+p}\}_{p=1}^{P}$, consider the set of all possible (α,μ)-constructible sequences which according

to Corollary 43 are:

$$\begin{aligned}\lambda(\alpha,\mu) :=& \{\{\lambda_m\}_{m=1}^M \in \operatorname{adm}(\alpha) : \{\lambda_m\}_{m=1}^M \text{ is } (\alpha,\mu)\text{-constructible}\}\\ =& \{\{\lambda_m\}_{m=1}^M \in \operatorname{adm}(\alpha) : \{\mathrm{DS}(\Lambda)_j\}_{j=1}^M \succeq \{\mu_{N+p}\}_{p=1}^P\}.\end{aligned}$$

We begin in Section 6.1 with a discussion on frame metrics. In particular we derive the MSE and FP and then expand our measure of optimality to include majorization. Then in Section 6.2, we present a partial solution to Problem 44—an explicit formula for the optimal (α,μ)-constructible sequence in the special case when all $\{\mu_{N+p}\}_{p=1}^P$ are of equal lengths.

6.1 Frame metrics

Errors are introduced when a signal, being transmitted over a communications networks, is distorted by noise or when part of it is lost due to network errors. In such a situation, the original signal x must be reconstructed from $F^*x+\epsilon$ where $\epsilon = [\epsilon_1\, \epsilon_2 \ldots \epsilon_N]^{\mathrm{T}} = \sum_{n=1}^N \epsilon_n e_n$, is the zero-mean, independent, identically distributed added noise with variance σ^2. In order to approximate the original signal, we apply the *Moore-Penrose generalized inverse* or *pseudoinverse* of F^*. Therefore, the reconstructed signal becomes $(FF^*)^{-1}F(F^*x+\epsilon)$ which has error given by $(FF^*)^{-1}F\epsilon$. In order to find the best reconstruction, we desire for this error to be a minimum. Specifically, we integrate the norm squared of the error $\|(FF^*)^{-1}F\epsilon\|^2$ over $\mathbb{R}^N$ to derive the MSE:

$$\mathrm{MSE} = \int_{\mathbb{R}^N} \|(FF^*)^{-1}F\Big(\sum_{n=1}^N \epsilon_n e_n\Big)\|^2 p_1(\epsilon_1)p_2(\epsilon_2)\ldots p_N(\epsilon_N)\mathrm{d}\epsilon_1\mathrm{d}\epsilon_2\ldots\mathrm{d}\epsilon_N. \tag{6.1}$$

Here, $p_n(\epsilon_n)$ is the probability density function for ϵ_n. Writing $p(\epsilon) = p_1(\epsilon_1)\dots p_N(\epsilon_N)$ in (6.1) gives,

$$\begin{aligned}
\text{MSE} &= \int_{\mathbb{R}^N} \Big\langle (FF^*)^{-1}F\Big(\sum_{n=1}^{N}\epsilon_n e_n\Big), (FF^*)^{-1}F\Big(\sum_{n=1}^{N}\epsilon_n e_n\Big)\Big\rangle p(\epsilon)\mathrm{d}\epsilon \\
&= \int_{\mathbb{R}^N} \sum_{n=1}^{N}\sum_{n'=1}^{N} \epsilon_n\epsilon_{n'}\langle (FF^*)^{-1}Fe_n, (FF^*)^{-1}Fe_{n'}\rangle p(\epsilon)\mathrm{d}\epsilon \\
&= \sum_{n=1}^{N}\sum_{n'=1}^{N}\langle (FF^*)^{-1}Fe_n, (FF^*)^{-1}Fe_{n'}\rangle \int_{\mathbb{R}^N}\epsilon_n\epsilon_{n'}p(\epsilon)\mathrm{d}\epsilon. \qquad (6.2)
\end{aligned}$$

Focusing on the integral of (6.2), and using the fact that each ϵ_n is an independent and identically distributed random variable with mean zero and variance σ^2 we see that:

$$\begin{aligned}
\int_{\mathbb{R}^N}\epsilon_n\epsilon_{n'}p(\epsilon)\mathrm{d}\epsilon &= \int_{\mathbb{R}^N}\epsilon_n\epsilon_{n'}p_1(\epsilon_1)p_2(\epsilon_2)\dots p_N(\epsilon_N)\mathrm{d}\epsilon_1\mathrm{d}\epsilon_2\dots\mathrm{d}\epsilon_N \\
&= \begin{cases} \int_{-\infty}^{\infty}\epsilon_n^2 p_n(\epsilon_n)\mathrm{d}\epsilon_n & n = n' \\ \Big(\int_{-\infty}^{\infty}\epsilon_n p_n(\epsilon_n)\mathrm{d}\epsilon_n\Big)\Big(\int_{-\infty}^{\infty}\epsilon_{n'} p_{n'}(\epsilon_{n'})\mathrm{d}\epsilon_{n'}\Big) & n \neq n' \end{cases} \\
&= \begin{cases} \mathrm{var}(\epsilon_n) & n = n' \\ 0 & n \neq n' \end{cases}. \qquad (6.3)
\end{aligned}$$

Substituting (6.3) into (6.2) gives:

$$\begin{aligned}
\text{MSE} &= \sum_{n=1}^{N}\sum_{n'=1}^{N}\langle (FF^*)^{-1}Fe_n, (FF^*)^{-1}Fe_{n'}\rangle \int_{\mathbb{R}^N}\epsilon_n\epsilon_{n'}p(\epsilon)\mathrm{d}\epsilon \\
&= \sum_{n=1}^{N}\sum_{n'=1}^{N}\langle F(FF^*)^{-2}Fe_n, e_{n'}\rangle \begin{cases} \epsilon^2 & n = n' \\ 0 & n \neq n' \end{cases} \\
&= \sigma^2\sum_{n=1}^{N}\langle F(FF^*)^{-2}Fe_n, e_n\rangle \\
&= \sigma^2\mathrm{Tr}(F(FF^*)^{-2}F) \\
&= \sigma^2\mathrm{Tr}((FF^*)^{-1}) \\
&= \sigma^2\sum_{m=1}^{M}\frac{1}{\lambda_m}, \qquad (6.4)
\end{aligned}$$

where $\{\lambda_m\}_{m=1}^M$ are the eigenvalues of FF^*. Since calculating the MSE involves inverting FF^*, which can be difficult or impossible, another metric that may be used is the FP. The FP is a measure of the total orthogonality of the frame vectors [5]. Inspired by Columb's Law, the FP involves visualizing the movement of M charged particles restricted to M concentric spheres [9]. The FP of a sequence $\{f_n\}_{n=1}^N$ in $\mathbb{R}^M$ is given by:

$$\mathrm{FP}(\{f_n\}_{n=1}^N) = \sum_{n=1}^N \sum_{n'=1}^N |\langle f_n, f_{n'}\rangle|^2. \tag{6.5}$$

Note that $\langle f_n, f_{n'}\rangle$ is the (n,n')th entry of the Gram matrix F^*F, and so the FP is an L^2 norm squared on the entries of F^*F. Recalling the Frobenius norm of a matrix A defined by $\|A\|_F^2 = \mathrm{Tr}(A^*A)$, (6.5) becomes,

$$\mathrm{FP}(\{f_n\}_{n=1}^N) = \sum_{n=1}^N \sum_{n'=1}^N |\langle f_n, f_{n'}\rangle|^2 = \sum_{n=1}^N \sum_{n'=1}^N |(F^*F)_{n,n'}|^2 = \|F^*F\|_F^2 = \mathrm{Tr}((FF^*)^2) = \sum_{m=1}^M \lambda_m^2. \tag{6.6}$$

Minimizing this quantity involves finding a sequence $\{f_n\}_{n=1}^N$ whose elements are as orthogonal to each other as possible [35].

Note that the MSE and FP are of a similar form which can be generalized to a family of convex functionals. A Schur-convex function is one which preserves the ordering of majorization, that is, $\phi(x) \geq \phi(y)$ whenever $x \geq y$. The following proposition, a combination of results from [30] and [42], will be useful in determining the optimal (α, μ)-constructible sequence.

Proposition 46 (Proposition 3.C.1. of [38])**.** *If $I \subset \mathbb{R}$ is an interval and $g : I \to \mathbb{R}$ is convex, then*

$$\phi(x) = \sum_{m=1}^M g(x_m),$$

is Schur-convex on I^M. Consequently, $x \geq y$ on I^M implies $\phi(x) \geq \phi(y)$.

If $g(x) = x^2$, then $\phi(x)$ becomes the FP (6.6), and if $g(x) = \frac{1}{x}$ then $\phi(x)$ becomes the MSE (6.4). The exponential function, $\phi(x) = \sum_{m=1}^M e^{x_m}$, is yet another example of a Schur-convex function since $g(x) = e^x$ is strictly convex. Moreover, the converse of

Proposition 46 is also true: if $\sum_{m=1}^{M} g(x_m) \geq \sum_{m=1}^{M} g(y_m)$ holds for all continuous convex functions g then $x \succeq y$ (Proposition 4.B.1 of [38]). In light of these facts, it suffices to show that the optimal (α, μ)-constructible sequence is the one which is majorized by all other sequences in $\lambda(\alpha, \mu)$ (Definition 45). Specifically, we say that a sequence $\{\tilde{\lambda}_m\}_{m=1}^{M} \in \lambda(\alpha, \mu)$ is the *optimal* (α, μ)-constructible sequence if $\{\tilde{\lambda}_m\}_{m=1}^{M} \preceq \{\lambda_m\}_{m=1}^{M}$ for all $\{\lambda_m\}_{m=1}^{M} \in \lambda(\alpha, \mu)$; that is, if

$$\sum_{m=1}^{k} \tilde{\lambda}_m \leq \sum_{m=1}^{k} \lambda_m, \quad \forall k = 1, \ldots, M-1,$$
$$\sum_{m=1}^{M} \tilde{\lambda}_m = \sum_{m=1}^{M} \lambda_m.$$

In the next section, we consider a special case when all $\{\mu_{N+p}\}_{p=1}^{P}$ are of equal lengths. In this case, we provide an explicit formula for the optimal (α, μ)-constructible sequence and verify that it is indeed optimal by showing that it is majorized by all other (α, μ)-constructible sequences in $\lambda(\alpha, \mu)$.

6.2 Construction of the optimal (α, μ)-constructible sequence

In this section, we construct the optimal (α, μ)-constructible sequence in the case when all $\{\mu_{N+p}\}_{p=1}^{P}$ are of equal lengths. We begin by considering what it means for a sequence $\{\lambda_m\}_{m=1}^{M}$ to be (α, μ)-constructible in this special case, and illustrate how to find the optimal (α, μ)-constructible sequence with an example. The main result of this section, Theorem 48, provides an explicit formula for the optimal (α, μ)-constructible whenever $\{\mu_{N+p}\}_{p=1}^{P}$ are of equal lengths.

To begin, recall from Chapter 5 that a sequence $\{\lambda_m\}_{m=1}^{M}$ is (α, μ)-constructible if and only if $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^{M} \succeq \{\mu_{N+p}\}_{p=1}^{P}$. When $\{\mu_{N+p}\}_{p=1}^{P}$ are of equal lengths, this majorization requirement can be simplified. In this case, the requirement that $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^{M} \succeq \{\mu_{N+p}\}_{p=1}^{P}$ comes for free since any sequence of diagonal sums $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^{M}$ majorizes the uniform sequence $\{\mu_{N+p}\}_{p=1}^{P}$ provided it has the proper sum (see discussion preceding Theorem 20). As such, to verify that $\{\lambda_m\}_{m=1}^{M}$ is (α, μ)-constructible, we need only check that it is

nonincreasing and that,

$$\sum_{p=1}^{P}\mu_{N+p}=\sum_{m=1}^{M}(\lambda_m-\alpha_m)=\sum_{j=1}^{M}\mathrm{DS}(\Lambda)_j. \tag{6.7}$$

Moreover, we claim that when $\{\mu_{N+p}\}_{p=1}^{P}$ are of equal lengths, any (α,μ)-constructible sequence $\{\lambda_m\}_{m=1}^{M}$ must satisfy:

$$\alpha_m\le\lambda_m\le\alpha_{m-P}\quad\forall m=1,\ldots,M, \tag{6.8}$$

where $\alpha_m:=\infty$ for $m\le 0$. The left hand side of (6.8) follows immediately from the fact that any (α,μ)-constructible sequence is necessarily α-admissible, i.e., $\alpha_m\le\lambda_m$ for all $m=1,\ldots,M$. The right hand side of (6.8) follows from the fact that subsequent sequence of continued outer eigensteps generated by $\{\lambda_m\}_{m=1}^{M}\in\lambda(\alpha,\mu)$ must interlace:

$$\lambda_m=\lambda_{N+P;m}\le\lambda_{N+P-1;m-1}\le\lambda_{N+P-2;m-2}\le\cdots\le\lambda_{N+1;m-(P-1)}\le\lambda_{N;m-P}=\alpha_{m-P}.$$

It turns out that if all of the added lengths are equal, any sequence which satisfies both (6.7) and (6.8) will automatically be (α,μ)-constructible, that is, $\{\mathrm{DS}(\Lambda)_j\}_{j=1}^{M}\succeq\{\mu_{N+p}\}_{p=1}^{P}$. This is not true if $\{\mu_{N+p}\}_{p=1}^{P}$ are of different lengths. For example, if $\{\mu_{N+1},\mu_{N+2}\}=\{\frac{1}{2},\frac{5}{2}\}$, $\{\alpha_1,\alpha_2,\alpha_3\}=\{1,2,3\}$, and $\{\lambda_1,\lambda_2,\lambda_3\}=\{3,3,3\}$, $\sum_{m=1}^{3}(\lambda_m-\alpha_m)=3$ as required by (6.7) and (6.8) is satisfied for every λ_m, but $\{\lambda_m\}_{m=1}^{3}$ is not (α,μ)-constructible—specifically, $\frac{5}{2}\not\leq\mathrm{DS}(\Lambda)_1=2$. However, if the same spectrum was built by $\mu_{N+1}=\mu_{N+2}=\frac{3}{2}$, $\{\lambda_m\}_{m=1}^{3}$ will automatically be (α,μ)-constructible since $\frac{3}{2}\le\mathrm{DS}(\Lambda)_1=2$ and $3=\mathrm{DS}(\Lambda)_1+\mathrm{DS}(\Lambda)_2=2+1=3$. In light of these facts, we now turn to an example illustrating how to find to optimal (α,μ)-constructible sequence.

Example 47. Let $M=3$ and $P=2$. In this example, we build the optimal (α,μ)-constructible sequence $\{\lambda_m\}_{m=1}^{3}$ where $\alpha=\{\alpha_1,\alpha_2,\alpha_3\}=\{\frac{7}{4},\frac{3}{4},\frac{1}{2}\}$ and $\mu=\{\mu_{N+1},\mu_{N+2}\}=\{2,2\}$. The initial sequence α is outlined in black in Figure 6.1(a). Just as in previous chapters, we can view the process of building the optimal (α,μ)-constructible sequence as

iteratively building a staircase. By looking at the quantities we wish to minimize, we can gain insight into what the optimal staircase should look like. For example, if we consider the FP, since this quantity is calculated by taking the sum of the squares of the eigenvalues $\{\lambda_m\}_{m=1}^3$, we would want to make the highest steps in the staircase as large as possible; that is, we would want to make λ_3 as large as possible, followed by λ_2, and finally λ_1. Similarly, if we consider the MSE, we would want to make the lowest steps in the staircase as small as possible. We know that there are limits on how large each step can be as well. In particular, since $\mu_{N+1} = \mu_{N+2} = 2$, each λ_m must satisfy (6.8) for all $m = 1, \ldots, 3$, specifically,

$$\begin{aligned} \tfrac{7}{4} = \alpha_1 &\leq \lambda_1 \leq \alpha_{-1} = \infty, \\ \tfrac{3}{4} = \alpha_2 &\leq \lambda_2 \leq \alpha_0 = \infty, \\ \tfrac{1}{2} = \alpha_3 &\leq \lambda_3 \leq \alpha_1 = \tfrac{7}{4}. \end{aligned} \tag{6.9}$$

The upper bound on λ_m is indicated by the dotted line in Figure 6.1(b).

To build the optimal (α, μ)-constructible sequence, we use a sort of "water-filling" approach. Water, representing the total amount of area that is added to the initial spectrum α–in this case $\mu_{N+1} + \mu_{N+2} = 4$ units of area—flows from left to right in the staircase picture. The water flows at a constant rate and will reach a barrier at the dotted lines whenever the value of λ_m is maximized. Water-filling stops whenever the total volume of water has be used. The optimal spectrum for α and μ chosen in this example is shown in Figure 6.1(c). The optimal (α, μ)-constructible sequence in this case is $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{21}{8}, \frac{21}{8}, \frac{7}{4}\}$.

We claim that this water-filling process will always produce the optimal (α, μ)-constructible sequence in the case that $\{\mu_{N+p}\}_{p=1}^P$ are all of equal lengths. In the following theorem, we provide an explicit formula for this optimal water-filled spectrum and prove that it is indeed optimal by showing that it is majorized by all other sequence $\{\lambda_m\}_{m=1}^M \in \lambda(\alpha, \mu)$.

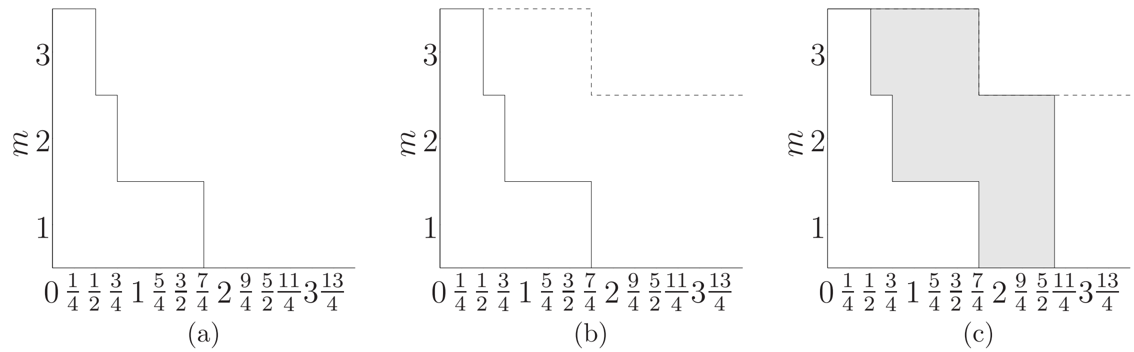

Figure 6.1: Given existing spectrum $\alpha = \{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$ (a), we add two additional vectors whose lengths are given by $\{\mu_{N+1}, \mu_{N+2}\} = \{2, 2\}$ so that the resulting spectrum is the optimal (α, μ)-constructible sequence. Since $\mu_{N+1} = \mu_{N+2} = 2$, by (6.8), $\alpha_m \leq \lambda_m \leq \alpha_{m-2}$ for all $m = 1, 2, 3$. The upper bound on λ_m is indicated by the dotted line in (b). Since no dotted line is present at levels $m = 1$ and $m = 2$, the upper bound is infinite. Water-filling with $\mu_{N+1} + \mu_{N+2} = 4$ units of volume yields the optimal (α, μ)-constructible sequence $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{21}{8}, \frac{21}{8}, \frac{7}{4}\}$ indicated by the gray area in (c).

Theorem 48. *Let $\{\alpha_m\}_{m=1}^{M}$ and $\{\mu_{N+p}\}_{p=1}^{P}$ be nonnegative nonincreasing sequences and suppose $\tilde{\mu} = \frac{1}{P}\sum_{p=1}^{P}\mu_{N+p}$. For $m = 1, \ldots, M$, define,*

$$V_m := (\alpha_{m-1} - \alpha_m)\min\{M - m + 1, P\}, \tag{6.10}$$

$$W_m := \sum_{k=m}^{M} V_k = \sum_{k=m}^{M}(\alpha_{k-1} - \alpha_k)\min\{M - k + 1, P\}, \tag{6.11}$$

where $\alpha_0 := \infty$. Define $\{\tilde{\lambda}_m\}_{m=1}^{M}$ according to the following rule: Pick any j such that $W_{j+1} \leq P\tilde{\mu} \leq W_j$, define:

$$\tilde{\lambda}_m := \begin{cases} \alpha_m, & 1 \leq m \leq j-1, \\ \alpha_j + \frac{1}{\min\{M-j+1,P\}}\left(P\tilde{\mu} - W_{j+1}\right), & j \leq m \leq j + \min\{M - j + 1, P\} - 1, \\ \alpha_{m-P}, & j + \min\{M - j + 1, P\} \leq m \leq M. \end{cases} \tag{6.12}$$

Then $\{\tilde{\lambda}_m\}_{m=1}^{M} \in \lambda(\alpha, \mu)$ and $\{\tilde{\lambda}_m\}_{m=1}^{M}$ is the optimal (α, μ)-constructible sequence, that is, $\{\tilde{\lambda}_m\}_{m=1}^{M} \preceq \{\lambda_m\}_{m=1}^{M}$ for all $\{\lambda_m\}_{m=1}^{M} \in \lambda(\alpha, \mu)$.

Proof. To prove that (6.12) is optimal, we must show that it is (α, μ)-constructible and that it is majorized by every sequence $\{\lambda_m\}_{m=1}^{M}$ in $\lambda(\alpha, \mu)$. We begin by showing that (6.12) is

(α, μ)-constructible, i.e., $\{\mathrm{DS}(\tilde{\Lambda})_j\}_{j=1}^M \succeq \{\mu_{N+p}\}_{p=1}^P$. By assumption $\{\mu_{N+p}\}_{p=1}^P$ are all of equal lengths, so in order for $\{\mathrm{DS}(\tilde{\Lambda})_j\}_{j=1}^M \succeq \{\mu_{N+p}\}_{p=1}^P$, it suffices to show that $\sum_{j=1}^M \mathrm{DS}(\tilde{\Lambda})_j = P\tilde{\mu}$. To do this, we first show that $\sum_{m=1}^M \tilde{\lambda}_m = \sum_{m=1}^M \lambda_m$. Summing (6.12) for all M:

$$\begin{aligned}\sum_{m=1}^M \tilde{\lambda}_m &= \sum_{m=1}^{j-1} \alpha_m + \sum_{m=j}^{j+\min\{M-j+1,P\}-1} \left[\alpha_j + \frac{1}{\min\{M-j+1,P\}}(P\tilde{\mu} - W_{j+1})\right] + \sum_{m=j+\min\{M-j+1,P\}}^{M} \alpha_{m-P} \\ &= \sum_{m=1}^{j-1} \alpha_m + \min\{M-j+1,P\}\alpha_j + P\tilde{\mu} - W_{j+1} + \sum_{m=j+\min\{M-j+1,P\}}^{M} \alpha_{m-P} \qquad (6.13)\end{aligned}$$

Substituting the fact that $P\tilde{\mu} = \sum_{p=1}^P \mu_{N+p} = \sum_{m=1}^M (\lambda_m - \alpha_m)$ for any arbitrary sequence $\{\lambda_m\}_{m=1}^M \in \lambda(\alpha, \mu)$, (6.13) becomes:

$$\begin{aligned}\sum_{m=1}^M \tilde{\lambda}_m &= \sum_{m=1}^M (\lambda_m - \alpha_m) + \sum_{m=1}^{j-1} \alpha_m + \sum_{m=j+\min\{M-j+1,P\}}^{M} \alpha_{m-P} + \min\{M-j+1,P\}\alpha_j - W_{j+1} \\ &= \sum_{m=1}^M \lambda_m - \sum_{m=j}^M \alpha_m + \sum_{m=j+\min\{M-j+1,P\}}^{M} \alpha_{m-P} + \min\{M-j+1,P\}\alpha_j - W_{j+1}. \qquad (6.14)\end{aligned}$$

To further simplify (6.14), we consider two different cases. First, we consider the case when $\min\{M-j+1,P\} = M-j+1$. In this case, (6.14) can be simplified as follows:

$$\sum_{m=1}^M \tilde{\lambda}_m = \sum_{m=1}^M \lambda_m - \sum_{m=j}^M \alpha_m + (M-j+1)\alpha_j - W_{j+1}. \qquad (6.15)$$

Next, we substitute the definition of W_{j+1} from (6.11), at which point (6.15) becomes:

$$\begin{aligned}\sum_{m=1}^M \tilde{\lambda}_m &= \sum_{m=1}^M \lambda_m - \sum_{m=j}^M \alpha_m + (M-j+1)\alpha_j - \sum_{m=j+1}^M (\alpha_{m-1} - \alpha_m)(M-m+1) \\ &= \sum_{m=1}^M \lambda_m - \sum_{m=j}^M \alpha_m + (M-j+1)\alpha_j - (M-j)\alpha_j + \sum_{m=j+1}^M \alpha_m \\ &= \sum_{m=1}^M \lambda_m,\end{aligned}$$

and so $\sum_{m=1}^M \tilde{\lambda}_m = \sum_{m=1}^M \lambda_m$ in the case that $\min\{M-j+1,P\} = M-j+1$, as claimed. Next, we consider the case when $\min\{M-j+1,P\} = P$. In this case, (6.14) becomes

$$\sum_{m=1}^M \tilde{\lambda}_m = \sum_{m=1}^M \lambda_m - \sum_{m=j}^M \alpha_m + \sum_{m=j+P}^M \alpha_{m-P} + P\alpha_j - W_{j+1}. \qquad (6.16)$$

Again, substituting (6.11) for W_{j+1}, (6.16) becomes:

$$\sum_{m=1}^{M}\tilde{\lambda}_m = \sum_{m=1}^{M}\lambda_m - \sum_{m=M-P+1}^{M}\alpha_m + P\alpha_j - \sum_{m=j+1}^{M}(\alpha_{m-1}-\alpha_m)\min\{M-m+1,P\} \tag{6.17}$$

We can further simplify (6.17) by splitting the last summation into two depending on whether $\min\{M-m+1,P\} = M-m+1$ or $\min\{M-m+1,P\} = P$:

$$\sum_{m=1}^{M}\tilde{\lambda}_m = \sum_{m=1}^{M}\lambda_m - \sum_{m=M-P+1}^{M}\alpha_m + P\alpha_j - \sum_{m=j+1}^{M-P}(\alpha_{m-1}-\alpha_m)P - \sum_{m=M-P+1}^{M}(\alpha_{m-1}-\alpha_m)(M-m+1). \tag{6.18}$$

Notice in (6.18), that two of the four sums on the right hand side of the equation are telescoping sums and so simplifying (6.18) results in many terms that cancel each other out yielding:

$$\sum_{m=1}^{M}\tilde{\lambda}_m = \sum_{m=1}^{M}\lambda_m - \sum_{m=M-P+1}^{M}\alpha_m + P\alpha_j - P\alpha_j + P\alpha_{M-P} - P\alpha_{M-P} + \sum_{m=M-P+1}^{M}\alpha_m = \sum_{m=1}^{M}\lambda_m.$$

Hence, $\sum_{m=1}^{M}\tilde{\lambda}_m = \sum_{m=1}^{M}\lambda_m$ in the case that $\min\{M-j+1,P\} = P$. Having this result, we are now able to prove that (6.12) is (α,μ)-constructible. Since $\{\mu_{N+p}\}_{p=1}^{P}$ are all of equal lengths by assumption, in order for (6.12) to be (α,μ)-constructible, it suffices to show that $\sum_{j=1}^{M}\mathrm{DS}(\tilde{\Lambda})_j = P\tilde{\mu}$:

$$\sum_{j=1}^{M}\mathrm{DS}(\tilde{\Lambda})_j = \sum_{m=1}^{M}(\tilde{\lambda}_m-\alpha_m) = \sum_{m=1}^{M}(\lambda_m-\alpha_m) = \sum_{j=1}^{M}\mathrm{DS}(\Lambda)_j = \sum_{p=1}^{P}\mu_{N+p} = P\tilde{\mu},$$

so indeed, (6.12) is (α,μ)-constructible.

Finally, we claim that (6.12) is optimal. In order to prove this, we must show (6.12) is majorized by all other (α,μ)-constructible sequences in $\lambda(\alpha,\mu)$. That is,

$$\sum_{m=1}^{k}\lambda_m \geq \sum_{m=1}^{k}\tilde{\lambda}_m, \quad \forall k=1,\ldots,M-1, \tag{6.19}$$

$$\sum_{m=1}^{M}\lambda_m = \sum_{m=1}^{M}\tilde{\lambda}_m, \tag{6.20}$$

for all $\{\lambda_m\}_{m=1}^{M} \in \lambda(\alpha,\mu)$. We just finished proving that (6.20) holds in order for (6.12) to be (α,μ)-constructible, and so we are left to show that (6.19) holds for $k=1,\ldots,M-1$. For

$1 \leq k \leq j-1$, (6.19) follows immediately from (6.12) and the left hand side of (6.8):

$$\sum_{m=1}^{k} \tilde{\lambda}_m = \sum_{m=1}^{k} \alpha_m \leq \sum_{m=1}^{k} \lambda_m.$$

Similarly for $j + \min\{M - j + 1, P\} \leq k \leq M$, (6.19) follows from (6.12) and negating the right hand side of (6.8); that is, $-\alpha_{m-P} \leq -\lambda_m$ for all m:

$$\sum_{m=1}^{k} \tilde{\lambda}_m = \sum_{m=1}^{M} \tilde{\lambda}_m - \sum_{m=k+1}^{M} \tilde{\lambda}_m = \sum_{m=1}^{M} \lambda_m - \sum_{m=k+1}^{M} \alpha_{m-P} \leq \sum_{m=1}^{M} \lambda_m - \sum_{m=k+1}^{M} \lambda_m = \sum_{m=1}^{k} \lambda_m.$$

For the remaining case when $j \leq k \leq \min\{M - j + 1, P\} - 1$, we first claim that

$$\frac{(k-j+1)}{\min\{M-j+1, P\}}(P\tilde{\mu} - W_{j+1}) \leq P\tilde{\mu} - \sum_{m=j+1}^{M} (\lambda_m - \alpha_m) + \sum_{m=j+1}^{k} \lambda_m - (k-j)\alpha_j. \tag{6.21}$$

To show that (6.21) is in fact true, we consider two separate cases—when $\tilde{\lambda}_k \leq \lambda_k$ and $\tilde{\lambda}_k \geq \lambda_k$. For the case when $\tilde{\lambda}_k \leq \lambda_k$, first note for any $j \leq k \leq \min\{M - j + 1, P\} - 1$, (6.12) gives

$$\frac{(k-j+1)}{\min\{M-j+1, P\}}(P\tilde{\mu} - W_{j+1}) = \sum_{m=j}^{k} \tilde{\lambda}_m - (k-j+1)\alpha_j. \tag{6.22}$$

In order to simplify (6.22), note from (6.12) that all values of $\tilde{\lambda}_m$ are equal whenever $j \leq m \leq \min\{M - j + 1, P\} - 1$. Combining this with the case that $\tilde{\lambda}_k \leq \lambda_k$, we have

$$\tilde{\lambda}_j = \tilde{\lambda}_{j+1} = \cdots = \tilde{\lambda}_k \leq \lambda_k \leq \lambda_{k-1} \leq \cdots \leq \lambda_j,$$

at which point (6.22) becomes:

$$\begin{aligned}
\frac{(k-j+1)}{\min\{M-j+1, P\}}(P\tilde{\mu} - W_{j+1}) &\leq \sum_{m=j}^{k} \lambda_m - (k-j+1)\alpha_j \\
&= (\lambda_j - \alpha_j) + \sum_{j=j+1}^{k} \lambda_m - (k-j)\alpha_j \\
&\leq \sum_{m=1}^{j} (\lambda_j - \alpha_j) + \sum_{j=j+1}^{k} \lambda_m - (k-j)\alpha_j \\
&= P\tilde{\mu} - \sum_{m=j+1}^{M} (\lambda_m - \alpha_m) + \sum_{m=j+1}^{k} \lambda_m - (k-j)\alpha_j.
\end{aligned}$$

Thus, (6.21) holds in the case that $\tilde{\lambda}_k \leq \lambda_k$.

In the second case when $\tilde{\lambda}_k \geq \lambda_k$, first note that since (6.12) is (α, μ)-constructible and $\tilde{\lambda}_m = \alpha_m$ for $m = 1, \ldots, j-1$:

$$\sum_{m=1}^{M}(\tilde{\lambda}_m - \alpha_m) = \sum_{m=1}^{j-1}(\alpha_m - \alpha_m) + \sum_{m=j}^{M}(\tilde{\lambda}_m - \alpha_m) = \sum_{m=j}^{M}(\tilde{\lambda}_m - \alpha_m). \tag{6.23}$$

Substituting (6.23) into the right hand side of (6.21) gives

$$P\tilde{\mu} - \sum_{m=k+1}^{M} \lambda_m + \sum_{m=j+1}^{M} \alpha_m - (k-j)\alpha_j = \sum_{m=j}^{M}(\tilde{\lambda}_m - \alpha_m) - \sum_{m=k+1}^{M} \lambda_m + \sum_{m=j+1}^{M} \alpha_m - (k-j)\alpha_j. \tag{6.24}$$

Next, we use our assumption that $\tilde{\lambda}_k \geq \lambda_k$ and the fact that all values of $\tilde{\lambda}_m$ are equal whenever $j \leq m \leq \min\{M-j+1, P\} - 1$; specifically, this implies

$$\tilde{\lambda}_{j+\min\{M-j+1,P\}-1} = \cdots = \tilde{\lambda}_{k+1} = \tilde{\lambda}_k \geq \lambda_k \geq \lambda_{k+1} \cdots \geq \lambda_{j+\min\{M-j+1,P\}-1},$$

and so

$$\sum_{m=k+1}^{\min\{M-j+1,P\}-1} \tilde{\lambda}_m \geq \sum_{m=k+1}^{\min\{M-j+1,P\}-1} \lambda_m. \tag{6.25}$$

Combining the right hand side of (6.8) with the fact that $\tilde{\lambda}_m = \alpha_{m-P}$ for $m = \min\{M-j+1, P\}, \ldots, M$, we obtain a similar expression:

$$\sum_{m=\min\{M-j+1,P\}}^{M} \tilde{\lambda}_m = \sum_{m=\min\{M-j+1,P\}}^{M} \alpha_{M-P} \geq \sum_{m=\min\{M-j+1,P\}}^{M} \lambda_m. \tag{6.26}$$

Having (6.25) and (6.26), we continue (6.24):

$$\begin{aligned} P\tilde{\mu} - \sum_{m=k+1}^{M} \lambda_m + \sum_{m=j+1}^{M} \alpha_m - (k-j)\alpha_j &\geq \sum_{m=j}^{M}(\tilde{\lambda}_m - \alpha_m) - \sum_{m=k+1}^{M} \tilde{\lambda}_m + \sum_{m=j+1}^{M} \alpha_m - (k-j)\alpha_j \\ &= \sum_{m=j}^{k} \tilde{\lambda}_m - \sum_{m=j}^{M} \alpha_m + \sum_{m=j+1}^{M} \alpha_m - (k-j)\alpha_j \\ &= \sum_{m=j}^{k} \tilde{\lambda}_m - (k-j+1)\alpha_j. \end{aligned} \tag{6.27}$$

Finally, we substitute for $\tilde{\lambda}_m$ according to (6.12) in order to obtain (6.21):

$$\begin{aligned}
P\tilde{\mu} - \sum_{m=k+1}^{M} \lambda_m + \sum_{m=j+1}^{M} \alpha_m - (k-j)\alpha_j \\
&\geq (k-j+1)\left[\alpha_j + \frac{1}{\min\{M-j+1, P\}}(P\tilde{\mu} - W_{j+1})\right] - (k-j+1)\alpha_j \\
&= \frac{(k-j+1)}{\min\{M-j+1, P\}}(P\tilde{\mu} - W_{j+1}).
\end{aligned}$$

Having that (6.21) is indeed true, we now show (6.19) in the case that $j \leq k \leq \min\{M-j+1, P\} - 1$:

$$\begin{aligned}
\sum_{m=1}^{k} \tilde{\lambda}_m &= \sum_{m=1}^{j-1} \alpha_m + \sum_{m=j}^{k}\left[\alpha_j + \frac{1}{\min\{M-j+1, P\}}(P\tilde{\mu} - W_{j+1})\right] \\
&= \sum_{m=1}^{j-1} \alpha_m + (k-j+1)\alpha_j + \frac{(k-j+1)}{\min\{M-j+1, P\}}(P\tilde{\mu} - W_{j+1}) \\
&\leq \sum_{m=1}^{j} \alpha_m + P\tilde{\mu} - \sum_{m=j+1}^{M} (\lambda_m - \alpha_m) + \sum_{m=j+1}^{k} \lambda_m \\
&= \sum_{m=1}^{j} \alpha_m + \sum_{m=1}^{j} (\lambda_m - \alpha_m) + \sum_{m=j+1}^{k} \lambda_m \\
&= \sum_{m=1}^{j} \lambda_m.
\end{aligned}$$

Therefore, $\{\tilde{\lambda}_m\}_{m=1}^{M}$ is majorized by every sequence $\{\lambda_m\}_{m=1}^{M} \in \lambda(\alpha, \mu)$ and so it is the optimal (α, μ)-constructible sequence. □

We leave the construction of the optimal (α, μ)-constructible sequence for any arbitrary set of lengths $\{\mu_{N+p}\}_{p=1}^{P}$ for future work.

VII. Conclusions and Future Work

In this dissertation, we considered the problem of constructing every frame whose frame operator has a given spectrum and whose vectors have prescribed lengths. Regardless of building a frame from scratch or completing a frame from preexisting measurements, the solution to this problem involved a two-step process—first, picking a sequence of eigensteps and second, constructing the frame vectors one by one. We also considered the problem of finding the optimal frame completion, and were able to prove optimality in the case where the set of lengths are all equal. While we feel confident that we know intuitively what the solution should be in general, proving optimality in *all* cases has proved to be difficult.

This leads directly into our future work. When we consider optimally completing a frame for any arbitrary set of lengths, any solution we have proposed to date has been in the form of an algorithm. The problem with this method is that it does not lend itself well to the proof techniques that we currently use or to prove that the conjectured optimal sequence is indeed optimal. The authors of [40] have already proposed their own algorithms for solving for the optimal frame completion problem, but their results, like our own, are just conjectures at this point. As such, our future work revolves around formally proving the solution we propose. We do not include our solution algorithm for the optimal (α, μ)-constructible sequence here, rather we close with an example illustrating how it works.

Example 49. Our goal is to construct the optimal (α, μ)-constructible sequence for a frame obtained by adding $P = 4$ additional vectors to a set of preexisting vectors in $\mathbb{R}^3$. For this example, $\alpha = \{\alpha_1, \alpha_2, \alpha_3\} = \{\frac{7}{4}, \frac{3}{4}, \frac{1}{2}\}$, and the added vector lengths $\mu = \{\mu_{N+p}\}_{p=1}^{4}$ are given by $\{\mu_{N+1}, \mu_{N+2}, \mu_{N+3}\} = \{2, 1, \frac{1}{4}, \frac{1}{4}\}$—the same values used in Example 42. We know from Corollary 43, that a sequence is (α, μ)-constructible provided its diagonal sums majorizes the set of added lengths. This majorization requirement involves taking partial sums of

diagonal sums as defined in Definition 37. As such, we give a special name to these partial sums:

Definition 50. Let $\Lambda \in \mathbb{R}^{M\times(M+1)}$. Define the *triangular sum* function $\mathrm{TS} : \mathbb{R}^M \to \mathbb{R}^M$

$$[\mathrm{TS}(\Lambda)]_j := \sum_{k=j}^{M} \mathrm{DS}(\Lambda)_k. \tag{7.1}$$

For this example, we can use Definition 50, to reformulate the original majorization inequalities given by (5.24) and (5.25) in terms of these new triangle sums. Specifically, in order for $\{\lambda_m\}_{m=1}^3$ to be (α,μ)-constructible, i.e., $\{DS(\Lambda)_j\}_{j=1}^3 \succeq \{\mu_{N+p}\}_{p=1}^4$, the optimal spectrum we will build must satisfy the following:

$$\begin{aligned}\sum_{p'=m}^{4} \mu_{N+p'} &\geq \sum_{j=m}^{3} \mathrm{DS}(\Lambda)_j = \mathrm{TS}(\Lambda)_m,\\ \sum_{p'=1}^{4} \mu_{N+p'} &= \sum_{j=1}^{3} \mathrm{DS}(\Lambda)_j = \mathrm{TS}(\Lambda)_1,\end{aligned} \tag{7.2}$$

for $m = 1,\dots,3$. We refer to (7.2) as the triangle sum constraints throughout the rest of this example.

Again, we can visualize the process of constructing the optimal (α,μ)-constructible sequence by building a staircase just as we did in Examples 15 and 42, with α being the initial set of steps. To build $\{\lambda_m\}_{m=1}^3$, we use the same "water-filling" approach used in Example 47. Water, representing the total amount of area that is added to the initial spectrum α, flows from left to right in the staircase picture. The water flows at a constant rate and will reach a barrier if equality occurs in one of the triangle sum constraints. For our example, if $\mathrm{TS}(\Lambda)_2 = \sum_{p=2}^4 \mu_{N+p}$, $\mathrm{TS}(\Lambda)_2$ is "full," meaning water is barred from the second and third lower diagonals. A barrier is placed to prevent water from flowing into these diagonals, and water is restricted to filling in the first diagonal, $\mathrm{DS}(\Lambda)_1$, from that point forward. Water filling stops when the total area requirement has been met, that is $\sum_{m=1}^M(\lambda_m - \alpha_m) = \sum_{p=1}^P \mu_{N+p}$—particularly for this example, $\sum_{m=1}^3(\lambda_m - \alpha_m) = \sum_{p=1}^4 \mu_{N+p} = \frac{7}{2}$. In contrast to finding the optimal (α,μ)-constructible sequence when all

the added lengths are equal, we are unable to determine where the barriers (dotted lines in Figure 6.1(b)) should be placed a priori.

We now simulate this water-filling process by introducing variables into the optimal spectral partition matrix we wish to build. We begin with the spectral partition matrix for α and let $\Lambda_{3,3} = x$.

$$\Lambda = \begin{bmatrix} 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & x & \frac{1}{2} \end{bmatrix} \tag{7.3}$$

In order for (7.3) to be constructible, $\sum_{j=m}^{3} DS(\Lambda)_j \leq \sum_{p=m}^{4} \mu_{N+p}$ for $m = 1, \ldots, 3$, and (7.3) must also be α-admissible according to Definition 35. Specifically, $\Lambda_{m,n} \geq 0$ for all $m = 1, \ldots, 3$ and $n = 1, \ldots, 4$, and we much check (ii) that $\Lambda_{m,n} \leq \Lambda_{m-1,n}$ for all $m = 2, \ldots, 3$ and $n = 1, \ldots, 4$. In light of these facts, x must be chosen so that the following inequalities occur simultaneously:

$$\begin{cases} \mathrm{TS}(\Lambda)_1 = x \leq \frac{7}{2} \\ \mathrm{TS}(\Lambda)_2 = 0 \leq \frac{3}{2} \\ \mathrm{TS}(\Lambda)_3 = 0 \leq \frac{1}{2} \\ 0 \leq x \leq \frac{1}{4} \end{cases} \Rightarrow x = \frac{1}{4}$$

Substituting $x = \frac{1}{4}$ into (7.3), and tracking the area in each triangle sum, we have:

$$\Lambda = \begin{bmatrix} 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad \begin{matrix} \mathrm{TS}(\Lambda)_1 = \frac{1}{4} \leq \frac{7}{2} \\ \mathrm{TS}(\Lambda)_2 = 0 \leq \frac{3}{2} \\ \mathrm{TS}(\Lambda)_3 = 0 \leq \frac{1}{2} \end{matrix} \tag{7.4}$$

At this point, summing across the rows of Λ gives that $\lambda_1 = \frac{7}{4}$, $\lambda_2 = \frac{3}{2}$, and $\lambda_3 = \frac{3}{2}$. Since $\sum_{m=1}^{3}(\lambda_m - \alpha_m) = \frac{1}{4} \leq \frac{7}{2}$, the total area requirement has not been met. Also note that since none of the triangle inequalities in (7.4) have been saturated (i.e., no equalities),

no barriers have been placed and water filling continues from left to right in the staircase picture. Continuing this process, we now let $\Lambda_{2,2} = \Lambda_{3,2} = x$.

$$\Lambda = \begin{bmatrix} 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & x & \frac{1}{4} & \frac{1}{2} \\ 0 & x & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Again we require the resulting Λ to be constructible, which amounts to finding the value of x such that the following inequalities occur simultaneously:

$$\begin{cases} \mathrm{TS}(\Lambda)_1 = 2x + \frac{1}{4} \le \frac{7}{2} \\ \mathrm{TS}(\Lambda)_2 = x \le \frac{3}{2} \\ \mathrm{TS}(\Lambda)_3 = 0 \le \frac{1}{2} \\ 0 \le x \le 1 \end{cases} \Rightarrow x = 1$$

$$\Lambda = \begin{bmatrix} 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad \begin{matrix} \mathrm{TS}(\Lambda)_1 = \frac{9}{4} \le \frac{7}{2} \\ \mathrm{TS}(\Lambda)_2 = 1 \le \frac{3}{2} \\ \mathrm{TS}(\Lambda)_3 = 0 \le \frac{1}{2} \end{matrix} \tag{7.5}$$

Summing across the rows of Λ gives that $\lambda_1 = \lambda_2 = \lambda_3 = \frac{7}{4}$. The total amount of area that has been added at this point is $\sum_{m=1}^{3}(\lambda_m - \alpha_m) = \frac{9}{4} \le \frac{7}{2}$, which gives $\frac{5}{4}$ units of area left to water-fill with. Again, none of the triangle inequalities in (7.5) have been saturated, so water filling continues from left to right. Next, we let $\Lambda_{1,1} = \Lambda_{2,1} = \Lambda_{3,1} = x$.

$$\Lambda = \begin{bmatrix} x & 1 & \frac{1}{4} & \frac{1}{2} \\ x & 1 & \frac{1}{4} & \frac{1}{2} \\ x & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

The following inequalities must be satisfied in order for Λ to be constructible:

$$\begin{cases} \mathrm{TS}(\Lambda)_1 = 3x + \frac{9}{4} \le \frac{7}{2} \\ \mathrm{TS}(\Lambda)_2 = 2x + 1 \le \frac{3}{2} \\ \mathrm{TS}(\Lambda)_3 = x \le \frac{1}{2} \\ 0 \le x \end{cases} = \begin{cases} x \le \frac{5}{12} \\ x \le \frac{1}{4} \\ x \le \frac{1}{2} \\ 0 \le x \end{cases} \Rightarrow x = \frac{1}{4}$$

Here we see that the value of $x = \frac{1}{4}$ is a result of $\mathrm{TS}(\Lambda)_2$ reaching its maximum value of $\frac{3}{2}$.

$$\Lambda = \begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad \begin{matrix} \mathrm{TS}(\Lambda)_1 = 3 \le \frac{7}{2} \\ \mathrm{TS}(\Lambda)_2 = \frac{3}{2} \le \frac{3}{2} \\ \mathrm{TS}(\Lambda)_3 = \frac{1}{4} \le \frac{1}{2} \end{matrix} \tag{7.6}$$

Barriers are placed in (7.6), indicated by the shaded bands in Λ, to prevent water from flowing into diagonals two and three. The remaining $\frac{1}{2}$units of area left to water fill with is restricted to filling in the first row of Λ. As such, we let x be the amount of area that can be added to $\Lambda_{1,1}$.

$$\begin{bmatrix} \frac{1}{4} + x & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Choosing x so that the following inequalities are satisfied simultaneously,

$$\begin{cases} \mathrm{TS}(\Lambda)_1 = x + 3 \le \frac{7}{2} \\ \mathrm{TS}(\Lambda)_2 = \frac{3}{2} \le \frac{3}{2} \\ \mathrm{TS}(\Lambda)_3 = \frac{1}{4} \le \frac{1}{2} \\ 0 \le x \end{cases} \Rightarrow x = \frac{1}{2}$$

gives $x = \frac{1}{2}$ which is a result of the total sum requirement being met, i.e., $\mathrm{TS}(\Lambda)_1$ has reached its maximum value. The water filling is now complete and the resulting optimal

(α, μ)-constructible matrix has been found.

$$\Lambda = \begin{bmatrix} \frac{3}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad \begin{aligned} \mathrm{TS}(\Lambda)_1 &= \tfrac{7}{2} \leq \tfrac{7}{2} \\ \mathrm{TS}(\Lambda)_2 &= \tfrac{3}{2} \leq \tfrac{3}{2} \\ \mathrm{TS}(\Lambda)_3 &= \tfrac{1}{4} \leq \tfrac{1}{2} \end{aligned} \tag{7.7}$$

Notice that all of the diagonals of (7.7) are now shaded because $\mathrm{TS}(\Lambda)_1$ has been saturated. That is, $\sum_{m=1}^{3}(\lambda_m - \alpha_m) = \sum_{p=1}^{4} \mu_{N+p}$. Summing across the rows of (7.7) gives the optimal (α, μ)-constructible sequence to be:

$$\tilde{\lambda}_1 = \frac{5}{2},\ \tilde{\lambda}_2 = 2,\ \tilde{\lambda}_3 = 2.$$

We claim that this is the sequences that is majorized by all other sequences which can be built from α and μ. For example, if we compare this optimal (α, μ)-constructible sequence to a nonoptimal one such as the one given in Example 42, we see that

$$\begin{aligned} \frac{13}{2} = \lambda_1 + \lambda_2 + \lambda_3 = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 = \frac{13}{2}, \\ \frac{11}{2} = \lambda_1 + \lambda_2 \geq \tilde{\lambda}_1 + \tilde{\lambda}_2 = \frac{9}{2}, \\ \frac{13}{4} = \lambda_1 \geq \tilde{\lambda}_1 = \frac{5}{2}, \end{aligned}$$

and so $\{\frac{13}{4}, \frac{9}{4}, 1\} \succeq \{\frac{5}{2}, 2, 2\}$. We leave the proof that $\{\tilde{\lambda}_m\}_{m=1}^{3}$ is majorized by all $\{\lambda_m\}_{m=1}^{3} \in \lambda(\alpha, \mu)$ for future work.

Appendix: MATLAB code

The following MATLAB code implements the algorithm given in Theorem 7. For the sake of simplicity, $V_n = \mathrm{I}$ for all n. The following two functions must be placed in the same directory in order to execute the code.

The first function `constructU.m` implements Step B of Theorem 7. Here the output of the function is a slightly modified version of Steps B.4 and B.5. The function `constructU.m` returns $U_n^* f_{n+1}$ and $U_n^* U_{n+1}$. We recursively call `constructU.m` and then multiply the outputs at each iteration in order to calculate f_n and U_n for $n = 2, \ldots, N$. This step is accomplished by the function `constructFrame.m` which outputs the final sequence of vectors $F = \{f_n\}_{n=1}^N$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^M$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n.

```
function [U, Uf] = constructU(E1, E2)
% Description:  This function implements Steps B.1-5 of the algorithm to
%               explicitly construct any and all sequence of vectors whose
%               partial-frame operator spectra match the eigensteps chosen
%               in Step A. Here, we assume V1,...,Vn are the identity.
% Call:      [U, Uf] = constructU(E1, E2)
%                   E1 = Spectrum at (n)
%                   E2 = Spectrum at (n+1)
% Output:       U = U_(n)* U_(n+1)
%               Uf = U_(n)* f_(n+1)
% File:         constructU.m

%spectra must be row vectors and listed in decending order****************
if ~isrow(E1), E1=E1'; end
if ~isrow(E2), E2=E2'; end
E1 = sort(E1, 'descend');
E2 = sort(E2, 'descend');

M = length(E1);

%Find indices of unique elements (Step B.2)*******************************
R1 = E1; %Unique set of eigenvalues of E1
R2 = E2; %Unique set of eigenvalues of E2
for i = 1:M
    [tf, loc] = ismember(E1(i), R2);
```

```
    if tf == 1
        [tf,loclast]=ismember(E1(i), R1);
        R1(loclast) = -1;
        R2(loc)=-1;
    end
end

%Index sets of unique elements of E1 and E2, respectively.
I = find(R1 >=0);
J = find(R2 >=0);

M1 = length(I);
M2 = M-M1;

R1 = R1(I);
R2 = R2(J);

%Construct column and row vectors (Step B.3)******************************
for i = 1:M1
    P(i) = sqrt(-prod(R1(i)-R2)/prod(R1(i)-R1(find(R1~=R1(i)))));
    Q(i) = sqrt(prod(R2(i)-R1)/prod(R2(i)-R2(find(R2~=R2(i)))));
end

%Construct difference matrix
for i = 1:M1
    D(i,:) = 1./(R2-R1(i));
end
W =(P'*Q).*D;

%Create Block Diagonal Matrix
UU=blkdiag(W,eye(M2));

%Permute Row and Columns;
PRow = permMat(I,M);
PCol = permMat(J,M);

%Compute U and Uf (Modified Steps B.4 and B.5)****************************
U = PRow*UU*inv(PCol);
Uf=zeros(M,1);
Uf(I) = P;
%************************************************************************
%Define permutation matrix given the unique index set I
function P = permMat(I,M)
m = 1:M;
pi = [I setdiff(m,I)];
Id = eye(M);
```

```
P = Id(1:M,pi);

%*************************************************************************
function tf = isrow(E)
[rows,cols]=size(E);
if rows==1
    tf=1;
else
    tf=0;
end

function F = constructFrame(E,U1)
% Description:  This function implements Step B of the algorithm to
%               explicitly construct any and all sequence of vectors whose
%               partial-frame operator spectra match the eigensteps chosen
%               in Step A. Here, we assume V1,...,Vn are the identity.
% Call:      F = constructFrame(E, U1)
%                   E = Matrix of eigensteps
%                   U1 = Initial unitary matrix
% Output:       The frame, F.
% File:         constructFrame.m

UU(:,:,1) = U1;
U(:,:,1) = UU(:,:,1);
F(:,1) = U(:,1,1);
[M,N] = size(E);

for i = 2:N
    [UU(:,:,i), Uf(:,i)] = constructU(E(:,i-1),E(:,i));
    U(:,:,i) = eye(M);

    %Multiply matrices to find new U
    for j = 1:i, U(:,:,i) = U(:,:,i)*UU(:,:,j);end

    %Multiply to find new f
    Temp = eye(M);
    for j = 1:i-1; Temp = Temp*UU(:,:,j); end
    F(:,i) = Temp*Uf(:,i);
end
```

The following example reproduces the results of Example 8 where the eigensteps are given by (3.28). For the sake of simplicity, $U_1 = \mathrm{I}$ and $V_n = \mathrm{I}$ for $n = 1, \ldots, 4$.

```
>> E = [0 0 0 2/3 5/3;0 1/3 4/3 5/3 5/3;1 5/3 5/3 5/3 5/3]
```

```
E =

         0         0         0    0.6667    1.6667
         0    0.3333    1.3333    1.6667    1.6667
    1.0000    1.6667    1.6667    1.6667    1.6667

>> U1=eye(3)

U1 =

     1     0     0
     0     1     0
     0     0     1

>> F = constructFrame(E,U1)

F =

    1.0000    0.6667   -0.4082   -0.1667    0.1667
         0    0.7454    0.9129    0.3727   -0.3727
         0         0         0    0.9129    0.9129
```

The following MATLAB code constructs a sequence of continued outer eigensteps using the Chop Kill algorithm given in Theorem 40:

```
function eigensteps=CK(lambda,alpha,mu)
% Description:  This function implements the Chop Kill (CK)
%               Algorithm and creates a sequence of eigensteps for lambda.
% Call:      eigensteps = CK(lambda,alpha,mu)
%                   lambda = final spectrum
%                   alpha = initial spectrum
%                   mu = set of lengths
% Output:       table of eigensteps
% File:         GTK.m

%spectra must be row vectors and listed in decending order****************
M = length(alpha);
P = length(mu);
T(:,:,P) = table(lambda,alpha);
eigensteps = zeros(M,P+1);
eigensteps(:,P+1) = flipud(sum(table(lambda,alpha),2));
eigensteps(:,1) = flipud(sum(table(alpha,alpha),2));
for p = P:-1:2
   newT = T(:,:,p);
   newT(find(tril(newT)))=0;
```

```
nu(M,1)=mu(p);
for j = M:-1:1
    for k = M-j+1:-1:1
        if k==(M-j+1)
            delta = min(nu(j-1+k,k), T(j-1+k,k,p));
        else
            delta = min(nu(j-1+k,k), T(j-1+k,k,p)-T(j+k,k,p));
        end

        newT(j-1+k,k) = T(j-1+k,k,p)-delta;

        if k>1
            nu(j-2+k,k-1) = nu(j-1+k,k) - delta;
        else
            nu(M,M-j+2) = nu(j-1+k,k) - delta;
        end

    end
end
eigensteps(:,p) = flipud(sum(newT,2));
T(:,:,p-1) = newT;
end
```

Bibliography

[1] Antezana, J., P. Massey, M. Ruiz and D. Stojanoff. "The Schur-Horn theorem for operators and frames with prescribed norms and frame operator," *Illinois J. Math.*, 51:537–560 (2007).

[2] Arnold, B. C. "Majorization: Here, There and Everywhere," *Statistical Science*, 22:407–413 (2007).

[3] Batson, J., D. A. Spielman and N. Srivastava. "Twice-Ramanujan sparsifiers," to appear in *SIAM J. Comput.*, (2011).

[4] Bendel, R. B. and M. R. Mickey. "Population correlation matrices for sampling experiments," *Comm. Statist. Simulation Comput.*, 7:163–182 (1978).

[5] Benedetto, J. J. and M. Fickus. "Finite normalized tight frames," *Adv. Comput. Math.*, 18:357–385 (2003).

[6] Bodmann, B. G. and P. G. Casazza. "The road to equal-norm Parseval frames," *J. Funct. Anal.*, 258:397–420 (2010).

[7] Boley G. and G. H. Golub. "A survey of matrix inverse eigenvalue problems," *Inverse Problems*, 3:595–622 (1987).

[8] Cahill, J., M. Fickus, D. G. Mixon, M. J. Poteet and N. Strawn. "Constructing finite frames of a given spectrum and set of lengths," to appear in *Applied and Computational Harmonic Analysis*, (2012).

[9] Casazza, P. G., M. Fickus, J. Kovačević, M.T. Leon and J. C. Tremain. "Physical laws governing finite tight frames," in *Proceedings of the SPIE Conference on Wavelets: Applications in Signal and Image Processing X*, 371-380 (November 2003).

[10] Casazza, P. G., M. Fickus, J. Kovačević, M.T. Leon and J. C. Tremain. "A physical interpretation of tight frames," in *Harmonic Analysis and Applications: In Honor of John J. Benedetto*, Ed. C. Heil, Birkhäuser, Boston, 2006.

[11] Casazza, P. G., M. Fickus and D. G. Mixon. "Auto-tuning unit norm tight frames", to appear in *Appl. Comput. Harmon. Anal.*, (2011).

[12] Casazza, P. G., M. Fickus, D. G. Mixon, Y. Wang and Z. Zhou. "Constructing tight fusion frames," *Appl. Comput. Harmon. Anal.*, 30:175–187 (2011).

[13] Casazza, P. G. and J. Kovačević. "Equal-norm tight frames with erasures," *Adv. Comp. Math.*, 18:387–430 (2003).

[14] Casazza, P. G. and M. Leon. "Existence and construction of finite tight frames," *J. Comput. Appl. Math.*, 4:277–289 (2006).

[15] Chan, N. N. and K.-H. Li. "Diagonal elements and eigenvalues of a real symmetric matrix," *J. Math. Anal. Appl.*, 91:562–566 (1983).

[16] Chu, M. T. "Constructing a Hermitian matrix from its diagonal entries and eigenvalues," *SIAM J. Matrix Anal. Appl.*, 16:207–217 (1995).

[17] Chu, M. T. and G. H. Golub. "Structured inverse eigenvalue problems," *Acta Numerica*, 11:1–71 (2002).

[18] Davies, P. I. and N. J. Higham. "Numerically stable generation of correlation matrices and their factors," *BIT*, 40:640–651 (2000).

[19] Dhillon, I. S., R. W. Heath, M. A. Sustik and J. A. Tropp. "Generalized finite algorithms for constructing Hermitian matrices with prescribed diagonal and spectrum," *SIAM J. Matrix Anal. Appl.*, 27:61–71 (2005).

[20] Duffin, R. J. and A. C. Schaeffer. "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.*, 72:341–366 (1952).

[21] Dykema, K., D. Freeman, K. Kornelson, D. Larson, M. Ordower and E. Weber. "Ellipsoidal tight frames and projection decomposition of operators," *Illinois J. Math.*, 48:477–489 (2004).

[22] Dykema,K. and N. Strawn. "Manifold structure of spaces of spherical tight frames," *Int. J. Pure Appl. Math.*, 28:217–256 (2006).

[23] Feng, D. J., L. Wang and Y. Yang. "Generation of finite tight frames by Householder transformations," Advances in Computational Mathematics, 24:297–309 (2006).

[24] Fickus, Matthew, Dustin G. Mixon and Miriam J. Poteet. "Frame completions for optimally robust reconstruction," *Proceedings of SPIE*, 8138: 81380Q/1-8 (2011).

[25] Fickus, Matthew, Dustin G. Mixon, Miriam J. Poteet and Nate Strawn. "Constructing all self-adjoint matrices with prescribed spectrum and diagonal," submitted to: *SIAM Journal on Matrix Analysis and Applications*, (2011).

[26] Goyal, V. K., J. Kovačević and J. A. Kelner. "Quantized frame expansions with erasures," *Appl. Comput. Harmon. Anal.*, 10:203–233 (2001).

[27] Goyal, V. K., M. Vetterli and N. T. Thao. "Quantized overcomplete expansions in $\mathbb{R}^N$: Analysis, synthesis, and algorithms," *IEEE Trans. Inform. Theory*, 44:16–31 (1998).

[28] Haemers, W. H. "Interlacing eigenvalues and graphs," *Lin. Alg. Appl.*, 226-228:593-616 (1995).

[29] Hardy, G. H., J. E. Littlewood and G. Pólya. *Inequalities*. Cambridge: Cambridge University Press, 1934.

[30] Hardy, G. H., J. E. Littlewood and G. Pólya. "Some simple inequalities satisfied by convex functions," *Messenger of Mathematics*, 58:145-152 (1929).

[31] Holmes, R. B. and V. I. Paulsen. "Optimal frames for erasures," *Linear Algebra Appl.*, 377:31–51 (2004).

[32] Horn, A. "Doubly stochastic matrices and the diagonal of a rotation matrix," *Amer. J. Math.*, 76:620–630 (1954).

[33] Horn, R. A. and C. R. Johnson. *Matrix Analysis*. Cambridge: Cambridge University Press, 1985.

[34] Johnson, C. R. and H. A. Robinson. "Eigenvalue inequalities for principal submatrices," *Lin. Alg. Appl.*, 37:11-22 (1981).

[35] Kovačević, J. and A. Chebira. "Life beyond bases: The advent of frames (Part I)," *IEEE Signal Process. Mag.*, 24:86–104 (2007).

[36] Kovačević, J. and A. Chebira. "Life beyond bases: The advent of frames (Part II)," *IEEE Signal Process. Mag.* 24:115–125 (2007).

[37] Leite, R. S., T. R. W. Richa and C. Tomei. "Geometric proofs of some theorems of Schur-Horn type," *Linear Algebra Appl.*, 286:149–173 (1999)

[38] Marshall, O. A., I. Olkin and B. C. Arnold. *Inequalities: Theory of Majorization and Its Applications*, Springer, 2010.

[39] Massey, P. and M. Ruiz. "Tight frame completions with prescribed norms," *Sampl. Theory Signal Image Process.*, 7:1–13 (2008).

[40] Massey P., M. Ruiz and D. Stojanoff. "Optimal dual frames and frame completions for majorization", preprint arXiv:1108.4412 (2012).

[41] Massey P., M. Ruiz and D. Stojanoff. "Optimal completions of a frame", preprint arXiv:1206.3588v1 (2012).

[42] Schur, I. "Über eine klasse von mittelbildungen mit anwendungen auf die determinantentheorie," *Sitzungsber. Berl. Math. Ges.*, 22:9–20 (1923).

[43] Strawn, N. K. "Finite frame varieties: nonsingular points, tangent spaces, and explicit local parameterizations," to appear in *J. Fourier Anal. Appl.*, (2011).

[44] Tropp, J. A., I. S. Dhillon and R. W. Heath. "Finite-step algorithms for constructing optimal CDMA signature sequences," *IEEE Trans. Inform. Theory*, 50:2916–2921 (2004).

[45] Tropp, J. A., I. S. Dhillon, R. W. Heath and T. Strohmer. "Designing structured tight frames via an alternating projection method," *IEEE Trans. Inform. Theory*, 51:188–209 (2005).

[46] Viswanath, P. and V. Anantharam. "Optimal sequences and sum capacity of synchronous CDMA systems," *IEEE Trans. Inform. Theory*, 45:1984–1991 (1999).

[47] Waldron, S. "Generalized Welch bound equality sequences are tight frames," *IEEE Trans. Inform. Theory*, 49:2307–2309 (2003).

[48] Welch, L. "Lower bounds on the maximum cross correlation of signals," *IEEE Trans. Inform. Theory*, 20:397–399 (1974).

[49] Weyl, H. "Das asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichungen (mit einer anwendung auf die theorie der hohlraumstrahlung)," *Mathematische Annalen*, 71:441–479 (1912).

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704–0188*

1. REPORT DATE *(DD–MM–YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From — To)*
13–12–2012	Doctoral Dissertation	Oct 2009–Dec 2012

4. TITLE AND SUBTITLE
Parametrizing finite frames and optimal frame completions

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Poteet, Miriam J., Civilian

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Force Institute of Technology
Graduate School of Engineering and Management (AFIT/EN)
2950 Hobson Way
WPAFB, OH 45433-7765

8. PERFORMING ORGANIZATION REPORT NUMBER
AFIT-ENC-DAM-12-02

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Intentionally Left Blank

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT
DISTRIBUTION STATEMENT A: APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

13. SUPPLEMENTARY NOTES

14. ABSTRACT
Frames are used in many signal processing applications. We consider the problem of constructing every frame whose frame operator has a given spectrum and whose vectors have prescribed lengths. For a given spectrum and set of lengths, we know when such a frame exists by the Schur-Horn Theorem; it exists if and only if its spectrum majorizes its squared lengths. We provide a more constructive proof of Horn's original result. This proof is based on a new method for constructing any and all frames whose frame operator has a prescribed spectrum and whose vectors have prescribed lengths. Constructing all such frames requires one choose eigensteps—a sequence of interlacing spectra—which transform the trivial spectrum into the desired one. We give a complete characterization of the convex set of all eigensteps. Taken together, these results permit us, for the first time, to explicitly parametrize the set of all frames whose frame operator has a given spectrum and whose elements have a given set of lengths. Moreover, we generalize this theory to the problem of constructing optimal frame completions. That is, given a preexisting set of measurements, we add new measurements so that the final frame operator has a given spectrum and whose added vectors have prescribed lengths. We introduce a new matrix notation for representing the final spectrum with respect to the initial spectrum and prove that existence of such a frame relies upon a majorization constraint involving the final spectrum and the frame's matrix representation. In a special case, we provide a formula for constructing the optimal frame completion with respect to fusion metrics such as the mean square error (MSE) and frame potential (FP). Such fusion metrics provide a means of evaluating the efficacy of reconstructing signals which have been distorted by noise.

15. SUBJECT TERMS
frame construction, tight, unit norm, equal norm, interlacing, majorization, Schur-Horn

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Matthew Fickus, PhD, Matthew.Fickus@afit.edu
					19b. TELEPHONE NUMBER *(include area code)*
U	U	U	UU	159	(937) 255-3636 ext. 4513